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MARKTMODELLE UND MARKET MAKER SPREADS

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CHAPTER 1 Introduction

Abstract

The purpose of this thesis is to analyse market maker quoting behaviour at Eurex. A market maker quotes ask prices as well as bid prices, i.e. he or she is willing to sell for the ask price as well as buy for the bid price. We will look at bid and ask prices on equity index options and try to find explanatory variables for the cross-sectional distribution of the spread, i.e. the difference between the ask and the bid price. In contrast to former studies we have had the opportunity to analyse the quoting behaviour of individual market maker. Hence we can identify and ignore unwanted side effects of market maker models.
We could explain most of the cross-sectional distribution with a three factor model. However, the main influence was found to be different for each market maker. An analysis of three different market makers reveals the spread size between bid and ask prices of a call option strongly depends on:

- holding risk, measured by squared delta
- liquidity, measured by the time to expiry
- cost of a substitute for a call position, measured by the spread of an equivalent put option, i.e. a put option with the same strike and time to expiry as the call option.

Surprisingly there was no significant correlation between market maker spreads and volatility of volatility. A fourth market maker taken into considerations just maintained maximum spreads.

Zielsetzung dieser Diplomarbeit war es, das Quotier-Verhalten von *Market Makern* im Bereich Aktienindex-Optionen zu beschreiben und dadurch Faktoren zu eruieren, die die Größe der Spanne von Geld- zu Brief-Kursen, d. h. den Unterschied zwischen Kauf- und Verkaufspreis einer Option, beeinflußen. Im Gegensatz zu früheren Studien hatten wir die Möglichkeit das Quotier-Verhalten einzelner *Market Maker* zu untersuchen. Daher war es möglich, ungewollte Nebeneffekte wie modellspezifische Eigenarten zu identifizieren und aus der Analyse auszuschließen.
Ein Drei-Faktor-Modell konnte die meisten Beobachtungen erklären. Am Beispiel von drei verschiedenen *Market Makern* konnten wir zeigen, dass die Größe der Spanne zwischen Geld- und Brief-Kurs einer *Call*-Option primär abhängt von:

- dem Risiko eine offene Position zu halten, gemessen durch das quadrierte Delta der Option.
- der Liquidität, gemessen durch die Laufzeit der Option.
- den Kosten, eine *Call*-Option zu ersetzen, gemessen durch die Größe der Spanne einer *Put*-Option mit gleichem *Strike* und gleicher Laufzeit.

Der primäre Einflußfaktor war allerdings bei jedem der untersuchten *Market Maker* ein anderer. Ein Zusammenhang zwischen der Spanne von Geld- zu Brief-Kursen und der Volatilität der Volatilität war nicht erkennbar. Ein vierter *Market Maker* folgte der vorgegebenen maximalen Spanne.

Motivation

Understanding the quoting behaviour of market makers can offer guidance in improving and designing market making models. For example, competition between market makers will be increased by increasing the minimum tick size or decreasing maximum allowed spreads, which in turn will result in an increase in demand. However, if the maximum spread is too low, market makers will stop quoting, as they cannot cover their costs anymore or the risk they have to bear is not worth the marginal possible profit, eventually resulting in a decrease of supply. The function of an exchange is to provide a suitable market making model to create an equilibrium between supply and demand thereby ensuring high liquidity and transparency. Understanding the driving factors of market makers' spread sizes can offer guidance in how to set the minimum tick size or the maximum allowed spread size.

Das Verständnis des Quotier-Verhaltens verschiedener *Market Maker* kann Hilfe beim Entwickeln eines *Market Making*-Modells bieten. Beispielsweise wird durch Erhöhung der minimalen *Tick*-Größe oder durch Herabsetzung der maximal erlaubten Geld- zu Brief-Spanne die Konkurrenz zwischen *Market Makern* erhöht, was im Allgemeinen die Nachfrage steigert. Allerdings führt eine zu geringe maximal erlaubte Spanne dazu, dass *Market Maker* ihre Kosten nicht mehr decken können oder das Risiko eines möglichen Verlustes zu hoch einschätzen; beides verringert das Angebot.
Aufgabe einer Börse ist es, durch ein geeignetes *Market Making*-Modell ein Gleichgewicht zwischen Angebot und Nachfrage herzustellen und damit eine hohe Liquidität und Transparenz zu sichern. Eine Analyse der Einflußfaktoren auf die Größe der Geld- zu Brief-Spanne kann Hilfe beim Festlegen der minimalen *Tick*-Größe oder der maximal erlaubten Spanne bieten.

Overview

In the first part of this thesis we will introduce the mathematical background, especially option pricing under several model assumptions (chapter 2). We will start by defining a general market model in chapter 2.1, which is influenced by Hafner [51] and Etheridge [36]. Two main concepts of financial mathematics – arbitrage freeness and completeness – are explained in this context, loosely following Björk [10]. These definitions are derived from a real-world argumentation and are strongly connected to the number of specific equivalent measures (see definition 2.1.19). The exact relationship is the statement of the first and second fundamental theorem of asset pricing. The first one was developed by Kreps [62] and Harrison and Pliska [53] in the late seventies and further generalized by Delbaen and Schachermayer [27], Schachermayer [76], [77] and others. Following a proof form Björk [10] for

the second fundamental theorem we will see the dual character between the first and second fundamental theorem.
After these general thoughts we will present the Black-Scholes [11] model in the next chapter (2.2). Even though it is general knowledge that this model is flawed it is, due to its simplicity, probably still the most commonly used model for option pricing. We will comment on the main point of criticism of the Black-Scholes model – the existence of volatility smiles – in the next chapter. In this chapter we will also describe different definitions of volatility. Approaches to model volatility smiles are to consider a stochastic volatility process after Fouque et al. [46] (chapter 2.3) or to model the observable option prices, instead of the unobservable volatility process. The latter approach has been developed by Lyons [66] in 1997. These models are called "stochastic implied volatility models" and we will demonstrate one following Hafner [51] in chapter 2.4. The most important result was derived by Albanese et al. [2], who proved that the volatility process is defined by the option price process. After that we will give a brief overview of research done in market models with frictions in chapter 2.5. We will summarize the results of Leland [63] and Hoggard et al. [54] following Avellaneda and Parás [5] in a partial differential equation similar to the one derived in the standard Black-Scholes model. And use the model of Leland [63] to derive a upper and a lower border for call options. In this context we prove a put-call parity with different bid and ask prices, a similar result can be found in Gould and Galai [49].

The second part covers the analysis of cross-sectional distribution of market makers' spreads. We will start with an overview of the markets in question and present a brief summary of the current market making model at Eurex (chapter 3.1). To discern possible influence factors on the spread size, we will study costs and risks of market makers in chapter 3.2 and chapter 3.3. This was influenced by George and Longstaff [65] and Bollen et al. [13]. In the next chapter we will present available data structures (chapter 3.4), which we will use in the following chapter to verify our assumptions (chapter 3.5). After that we will run a linear regression between the spread size and our explanatory variables (chapter 3.6). For an introduction into data analysis we made use of Brachinger [15], Hamilton [52] and Enders [35]. A summary of our results and an outlook can be found in chapter 3.7.

> Um ein tieferes Verständnis für die Materie zu erhalten, werden wir uns im ersten Abschnitt dieser Diplomarbeit mit den finanz-mathematischen Hintergründen beschäftigen, insbesondere der Optionsbewertungs-Theorie (Kapitel 2).
> Wir beginnen damit, in Kapitel 2.1 ein allgemeines, von Hafner [51] und Etheridge [36] beeinflußtes Marktmodell zu definieren. In diesem Zusammenhang folgen wir Björk [10], um die beiden fundamentalen Konzepte Arbitrage-Freiheit und Vollständigkeit eines Marktes zu erläutern. Die aus der Praxis abgeleiteten Definitionen lassen sich in Zusammenhang mit der Anzahl gewisser äquivalenter Maße bringen (siehe Definition 2.1.19). Der präzise Zusammenhang zwischen der Anzahl äquivalenter Maße und den oben genannten Eigenschaften wird in zwei Fundamentalen Theoremen erläutert. Das erste Theorem geht ursprünglich auf Kreps [62] und Harrison & Pliska [53] zurück, wurde aber von Delbaen & Schachermayer [27], Schachermayer [76] [77] und anderen kontinuierlich weiterentwickelt. Im Beweis des zweiten Theorems wird der duale Charakter zwischen der Eigenschaft des Marktes frei von Arbitrage-Möglichkeiten zu sein und seiner Vollständigkeit sichtbar werden (Björk [10]). Nach den allgemeinen Vorüberlegungen werden wir dann in Kapitel 2.2 das Black-Scholes-Modell [11]

vorstellen, das sich durch seine Einfachheit auszeichnet und daher trotz bekannter Mängel immer noch das wohl weitverbreitetste Modell für die Optionsbewertung ist.
Im nächsten Abschnitt, Kapitel 2.3, werden wir die verschiedenen Begriffe der Volatilität vorstellen. Hierbei werden offensichtliche Mängel am Black-Scholes-Modell sichtbar. Eine mögliche Verbesserung des Black-Scholes-Modells führt zu stochastischen Volatilitäts-Modellen, in denen der Hauptkritikpunkt am Black-Scholes-Modell, nämlich die Annahme einer konstanten Volatilität, behoben wurde. Hierbei fassen wir die wesentlichen Punkte aus Fouque et al. [46] zusammen. Lyons [66] verfolgte 1997 einen anderen Ansatz, indem er die (beobachtbaren) Optionspreise anstelle der (unbeobachtbaren) Volatilität modellierte. Letzteres wurde durch die hohe Liquidität von Optionen ermöglicht. Solche Modelle werden stochastische implizierte Volatilitäts-Modelle genannt: das von Hafner [51] untersuchte Modell werden wir in Kapitel 2.4 vorstellen. Im Jahre 1998 wurde das wesentliche Resultat von Albanese et al. [2] bewiesen, nämlich dass sich die Volatilität der Aktie tatsächlich mittels geeigneter Optionspreise darstellen läßt.
Wir schliessen den theoretischen Teil ab mit einem Überblick verschiedener Studien über Marktmodelle mit Transaktionskosten. Wir werden die Resultate von Leland [63] und Hoggard et al. [54] zusammenfassen, was zu einer partiellen Differentialgleichung führt, die ähnlich der im Black-Scholes Modell hergeleiteten ist. Diese Zusammenfassung läßt sich in Avellaneda und Parás [5] finden. Anschliessend werden wir das Modell von Leland [63] nutzen, um eine obere und untere Schranke für den Preis einer *Call*-Option zu bestimmen. In diesem Zusammenhang beweisen wir eine *Put-Call*-Parität, was in ähnlicher Weise bei Gould und Galai [49] gefunden werden kann.

Im zweiten Teil dieser Diplomarbeit (Kapitel 3) werden wir uns dann insbesondere mit praktischen Feinheiten beschäftigen, um abschließend das tatsächliche Quotier-Verhalten verschiedener *Market Maker* zu untersuchen.
Zuerst werden wir die drei relevanten Aktienindizes vorstellen und danach das *Market Making*-Modell der Eurex präsentieren. Hierbei ist vor allem die von Eurex gesetzte maximale *Spread*-Größe von Wichtigkeit (Kapitel 3.1). Um mögliche Einflussfaktoren auf die *Spread*-Größe bestimmen zu können, analysieren wir in den Kapiteln 3.2 und 3.3 die verschiedenen Kostenfaktoren und Risiken der *Market Maker*. Die Vorgehensweise der Analyse wurde inspiriert von George & Longstaff [65] und Bollen et al. [13]. In Kapitel 3.4 werden die vorhandenen Datensätze vorgestellt, mit denen dann in Kapitel 3.5 die vorher gemachten Annahmen empirisch belegt werden. Abschließend werden wir in Kapitel 3.6 eine lineare Regression zwischen den *Spread*-Größen und den in den vorherigen Kapiteln gefundenen, möglichen Einflussfaktoren durchführen. Als mathematische Einführung in das Gebiet der Datenanalyse wurden Brachinger [15], Hamilton [52] und Enders [35] genutzt.
Eine Zusammenfassung der Ergebnisse des praktischen Teils dieser Diplomarbeit und einen Ausblick geben wir in Kapitel 3.7.

Approach

I started to work on my thesis on 1 February 2006 at Eurex (Deutsche Börse Frankfurt). The purpose of this thesis was to analyse a possible correlation between market maker spreads and the volatility of volatility or the volatility of a volatility index. First I had to collect all available data to calculate several volatility indices with different fixed time to expiry. Results showed frequent drops for the 90-day volatility index (figure 2.3.14). To explain this I had to understand the underlying mathematical theory, i.e. a variance swap (Carr and Wu [20]), and the exact definition of the several option series available at Eurex. I derived that these drops could be explained by the fact that volatility indices are only approximations of a variance swap.
I calculated the historical volatility of the volatility index and also the historical volatility of the historical volatility and compared it to the available market maker spreads. Unfortunately no correlation could be found. However, the dataset available was not appropriate for this analysis, because it was aggregated over all option series per day (see dataset 1). Also it seems questionable to calculate the volatility of volatility, with the standard formula, these rejections were fortified by Benth [8].
In March 2006 the way the data is stored was refined to only aggregate over each specific option series (see dataset 2). However, I could not use this data in more detail, because at the beginning the dataset was not comprehensive enough and at the end I had to take care to meet the deadline. Because of that I decided to use intraday data (dataset 3).

During my investigation of previous research done I came across George and Longstaff [65] who did a study of the bid/ask spread on S&P 100 index options and Bollen et al. [13] who analysed the bid/ask on stocks. Bollen et al. [13] used an option for measuring the risk. This inspired me to consider compound options as a possible risk measure. I derived that the compound option exactly nullifies the risk of losing from an open position. Hence, the price of a compound option can be thought of as a upper border for spread sizes. However, the price of a short-term compound option in the Black-Scholes model is correlated to the delta. Because I obtained strong correlation between delta and spread sizes, as was found by George and Longstaff [65], I rejected this approach. But instead I thought of pricing a compound option in a stochastic volatility model. This brought me to implied stochastic volatility models and especially Hafner [51]. Because completeness and arbitrage freeness are not comprehensively answered for these models, and both are substantial from a mathematical point of view, this motivated me to study these fundamental concepts in more detail. For this I frequently used Björk [10] as a basic reference. In a stochastic implied volatility model no-arbitrage arguments immediately lead to a correlation between volatility of volatility and the convexity of the implied volatility smile (Schönbucher [79]). This was especially appealing for me, because a correlation between the smile surface and the spread would induce a correlation between the volatility of volatility and the spread. However, the exact calculations of a smile surface is not easily done, especially to retrieve the exact interest rate is a significant difficulty.
George and Longstaff [65] stated that the spread of a put and a call option can be used to derive a theoretical spread on the underlying. After some fruitless attempts to retrieve this, I proved a put-call parity under transaction costs. A similar result was already derived by Gould and Galai [49] in 1974 and a more general version was used by Perrakis and Lefoll [71] in 2000.

For a summary of my results from the data analysis see above or chapter 3.7.1.

CHAPTER 2

Financial Market Models

In chapter 2.1 we will define a general continuous-time financial market model. The model is influenced by Hafner [51] and Etheridge [36]. The goal of this chapter is to explain assumptions necessary to retrieve a general pricing formula for derivative securities. And especially to retrieve a pricing formula for variance swaps. The market model will be used as a reference in chapter 2.2, where we set the volatility and drift process to be constant, resulting in the well known Black-Scholes [11] model. In chapter 2.3 we will present two standard definitions of volatility and later on give a brief introduction to stochastic volatility models, following Fouque et al. [46]. At the end of chapter 2.3 we will derive a term structure of the volatility of variance swaps (VDAX-NEW). The model defined in chapter 2.1 will also be used as a reference in chapter 2.4, where we model the stock price volatility – following Hafner [51] – indirectly through an observable implied volatility process. The fundamental basics for this approach are the Black-Scholes model and the implied volatility, which therefore had to be included for completeness. We will close the first part with presenting a pricing formula in a non-frictionless Black-Scholes model in chapter 2.5, following Avellaneda and Parás [5].

CHAPTER 2.1 General Financial Market Model

After introducing several basic mathematical terms and common conventions (2.1.1), which can be found in any textbook about stochastic analysis, for example Karatzas and Shreve [60] or Øksendal [69], we will define the market model used in this chapter (2.1.2). This model is influenced by Hafner [51], but instead of one single stock and infinitely many options, we allow a finite number of stocks and no options, as was done in Etheridge [36], pages 163 to 175. It will be used as a reference in the following chapters. In 2.1.3 we will define fundamental financial terms. These terms can be found in any financial textbook, e.g. Hull [55], Etheridge [36] or Hafner [51]. Loosely following Björk [10] we will present two highly desired characteristics of market models, namely completeness (2.1.5) and the lack of arbitrage opportunities (2.1.4). A simple and intuitive approach into arbitrage theory in a two state world is given by Etheridge [36]. For a more detailed approach the reader is referred to Kabanov [58]. It is sometimes common in non-mathematical textbooks to state that an arbitrage free market is equivalent to the existence of an equivalent martingale measure. However, this is not correct and we will prove the exact result following Schachermayer [76] and, for the other direction, Björk [10]. To emphasize the difference between the exact result and the "roughly speaking" result, we call it the "1. Fundamental Theorem of Asset Pricing (Strong)" and the "1. Fundamental Theorem of Asset Pricing (Weak)", respectively. Where the latter theorem is more or less an application of the Girsanov theorem together with the definition of the equivalent martingale measure, the theorem can be found in Øksendal [69]. To derive the relationship between complete markets and equivalent martingale measures, we will proof – following Björk [10] – an inverse Girsanov theorem, and then apply it to prove the "2. Fundamental Theorem of Asset Pricing".

The exact relationship between the financial term and a more abstract condition, used in the proof of the first and second fundamental theorem, are defined in lemmas 2.1.22 and 2.1.31, respectively. Because standard textbooks in general omit the fairly easy proof of lemma 2.1.22, we gave an exact proof of the relationship. The prove of lemma 2.1.31 can mainly be found in Björk [10] but had to be modified by a standard idea found in Karatzas and Shreve [60].

Standard proofs show that an equivalent martingale measure or a replicating portfolio allows us to derive pricing formulas for derivatives; this was influenced by Björk [10] (2.1.6).

Chapter 2.1.7 proves that the stock prices are lognormal distributed in a deterministic environment (exactly defined latter on) and using a result of Breeden and Litzenberger [16] that the risk-neutral distribution can be derived from the pricing formula in the general framework.

After that we try to find reasons for modelling the stock as we did in chapter 2.1.2. This argumentation is influenced by Osborne [70], Etheridge [36] and Hull [55].

Finally we use no-arbitrage arguments to get a pricing formula for put options, resulting in the well-known put-call parity. We also derive a pricing formula for variance swaps, where we combine proofs from Carr and Madan [21] and Hafner [51]. We close this chapter with a summary.

2.1.1 Basic Mathematical Terms

Let $T^* < \infty$ be our time horizon. We start with a filtered probability space $(\Omega, \mathcal{F}, \mathbf{P}, \mathbf{F})$, where Ω is the state space, $\mathcal{F} \subseteq \Omega$ a σ-algebra, $\mathbf{P} : \mathcal{F} \to [0, 1]$ the objective (real-world) probability measure and let $\mathbf{F} = (\mathcal{F}_t)_{t \in [0, T^*]}$ be the filtration generated by all 0-sets of $\mathbf{P}$ and an n-dimensional Brownian motion $(W_1, \ldots, W_n) = W : [0, T^*] \times \Omega \to \mathbb{R}^n$.[1] It therefore satisfies the usual conditions right continuity[2] and completeness. We further assume that $\mathcal{F}_{T^*} = \mathcal{F}$ and without loss of generality $\mathcal{F}_0 = \{\emptyset, \Omega\}$.

We only consider filtrations satisfying the usual conditions, because then right continuity implies:

$$\mathbf{P}(\forall t \in [0, \infty) \quad X_t = Y_t) = 1 \quad \Leftrightarrow \quad \forall t \in [0, \infty) \ \mathbf{P}(X_t = Y_t) = 1$$

and completeness implies for two stochastic processes X and Y satisfying either one of the above conditions:

$$X \ \mathbf{F}-\text{adapted} \ \Leftrightarrow \ Y \ \mathbf{F}-\text{adapted}$$

Definition 2.1.1 Set of Integrable Processes
The *set of integrable stochastic processes* $\mathbf{L}(X)$ on $(\Omega, \mathcal{F}, \mathbf{P}, \mathbf{F})$ with respect to an 1-dimensional (t-)continuous semimartingale $X(t, \omega) = X(0, \omega) + A(t, \omega) + M(t, \omega)$ – where M is the (uniquely defined) continuous local martingale and A is of finite variation[3] – is defined as the set of all predictable stochastic processes $\varphi : [0, T^*] \times \Omega \to \mathbb{R}$ with

$$\mathbf{P}\left(\omega \in \Omega \ \middle| \ \int_0^{T^*} \varphi^2(s, \omega) d{<}M{>}(s, \omega) < \infty\right) = 1 \tag{EQ 1}$$

$$\mathbf{P}\left(\omega \in \Omega \ \middle| \ \int_0^{T^*} |\varphi(s, \omega)| d|A|(s, \omega) < \infty\right) = 1 \tag{EQ 2}$$

Standard proofs show that for $\varphi \in \mathbf{L}(X)$ we can define the stochastic (Itô) integral:

$$(\varphi \bullet X)(t, \omega) := \int_0^t \varphi(s, \omega) dX(s, \omega).$$

If we have a d-dimensional continuous semimartingale $X_t = (X_{1,t}, \ldots, X_{d,t})$ and a d-dimensional process $\varphi_t = (\varphi_{1,t}, \ldots, \varphi_{d,t})$ with $\varphi_{i,t} \in \mathbf{L}(X_{i,t})$ for all $i \in \{1, \ldots, d\}$, we will write $\varphi \in \mathbf{L}^d(X)$ and $(\varphi \bullet X)(t, \omega)$ for $\sum_{i=1}^d (\varphi_i \bullet X_i)(t, \omega)$. In this context we usually just write $\varphi \in \mathbf{L}(X)$.

1. I.e. $\mathbf{F}$ is the augmentation of the filtration generated by a Brownian motion.
2. Every augmented filtration generated by a Brownian motion is right continuous, see Bauer [6] p. 483.
3. A process $A : [0, T^*] \times \Omega \to \mathbb{R}$ is called of finite variation, if the 1-variation up to time T^* as defined in footnote 1, p. 28, is finite, $\mathbf{P}$-almost surely. We wrote $|A|_t$ for ${}^1{<}A{>}_t$ in (EQ 2).

Definition 2.1.2 Itô Process
We call $X : [0, T^*] \times \Omega \to \mathbb{R}$ an *Itô process* on $(\Omega, \mathcal{F}, \mathbf{P}, \mathbf{F})$, if we can write

$$X(t, \omega) = X(0, \omega) + \int_0^t a(s, \omega)ds + \int_0^t b(s, \omega)dW(s, \omega) \qquad \text{(EQ 3)}$$

with X_0 being a $\mathcal{F}_0$-measurable process, W a standard Brownian motion (Wiener process) on $(\Omega, \mathcal{F}, \mathbf{P}, \mathbf{F})$, $b \in \mathbf{L}(W)$ and a a predictable processes satisfying the condition:

$$\forall t \in [0, T^*] \qquad \mathbf{P}\left(\omega \in \Omega \mid \int_0^t |a(s, \omega)|ds < \infty\right) = 1$$

so that the integrals in (EQ 3) are well-defined.

For convenience, equations like (EQ 3) are often written as

$$dX(t, \omega) = a(t, \omega)dt + b(t, \omega)dW(t, \omega)$$

It is also common practise to write $X_t(\omega)$ instead of $X(t, \omega)$ and to omit the realized state of the world ω for any random variable. Also we will always understand equality between two random variables X and Y as equality $\mathbf{P}$-almost surely, i.e.:

$$X = Y \Leftrightarrow \mathbf{P}(\omega \in \Omega \mid X(\omega) = Y(\omega)) = 1$$

Definition 2.1.3 Dolean's Exponential
Let X_t be a continuous local martingale we define *Dolean's exponential* $\mathcal{E}(X)$ by:

$$\forall t \in [0, T^*] \qquad \mathcal{E}(X)_t := \exp\left(-X_t - \frac{1}{2}<X>_t\right)$$

Using Itô's lemma[1] we can easily prove that $L_t := \mathcal{E}(X)_t$ satisfies $dL_t = (-L_t)dX_t \wedge L_0 = 1$.
It therefore follows that Dolean's exponential is a local martingale – in fact a supermartingale – in $(\Omega, \mathcal{F}, \mathbf{P}, \mathbf{F})$.

Definition 2.1.4 Novikov's Condition
Let W be a d-dimensional Brownian motion. We say a process $\varphi \in \mathbf{L}^d(W)$ satisfies *Novikov's condition*, if we have:

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^{T^*} \|\varphi_t\|^2 dt\right)\right] < \infty$$

It can be shown that if $\varphi \in \mathbf{L}^d(W)$ satisfies Novikov's condition, the Dolean exponential $\mathcal{E}(\varphi \bullet W)$ is a martingale.[2]

1. See appendix A.1.
2. See Karatzas and Shreve [60], p. 199.

2.1.2 Model Assumptions

Primary traded securities are a money marker account B_t and d-many non-dividend paying stocks $S_t = (S_{1,t}, \ldots, S_{d,t})$, also called assets or underlyings.

Assumption i) We have a frictionless security market[1], where market participants are allowed to trade continuously[2] up to a finite planning horizon T^*.

Assumption ii) The money market account (also called the numeraire) is modelled by formula 2.1.5 with constant interest rate $r \in \mathbb{R}$ and $B_0 = 1$. We will also write $S_{0,t}$ for B_t.

Formula 2.1.5 Money Market Account Process

$$\forall t \in [0, T^*]: \qquad dB_t = rB_t dt$$

Assumption iii) The stocks are modelled by formula 2.1.6 with positive, real-valued initial stock prices $S_{i,0} \in \mathbb{R}^+$ for all $i \in \{1, \ldots, d\}$, i.e. $S_0 \in \mathbb{R}_+^d$ and an invertible volatility matrix (ν_{ij}). Further we assume that S is an Itô process so that the stochastic differential equation (SDE) formula 2.1.6 is well defined. Also we assume that formula 2.1.6 has got a (strong) unique solution.[3]

Formula 2.1.6 Stock Price Process

$$\forall t \in [0, T^*]: \qquad dS_{i,t} = \mu_{i,t} S_{i,t} dt + \sum_{k=1}^{n} \nu_{ik}(t) S_{i,t} dW_{k,t}$$

Definition 2.1.7 Discounted Stock Price Process

The process defined by $S^* := S_0^{-1} S = (1, S_0^{-1} S_1, \ldots, S_0^{-1} S_d)$ is called the *discounted stock price process* from S.

Note that by definition the stock is continuously, which means we do not allow the stock to make jumps. By definition the volatility processes $\nu_{ij}(t)$ or the drift processes $\mu_{i,t}$ may be stochastic, which would imply, that the number of random sources n could be greater than the number of traded assets d.

We can easily see that the solution to formula 2.1.5 is given by $B_t = \exp(rt)$.

1. I.e. there are no bid/ask spreads, transaction fees, margin requirements or any restrictions on short sales. Also all assets have to be perfectly divisible, see [51], p. 10. All of these assumptions are normally not satisfied in reality. For example local regulations sometimes prohibit short sells, i.e. selling stocks which one does not own. Also a bid (sell price) and ask (buy price) spread can normally be found.
2. In reality continuous trading is of course unavailable, especially in illiquid markets.
3. For example we could assume that $x \cdot \mu_i(t, x)$ and $x \cdot \nu_{ik}(t, x)$ are global Lipschitz and that they satisfy a linear growth condition in x. For details see Karatzas and Shreve [60], p. 287 (uniqueness) and p. 289 (existence).

With the help of Itô's formula, we can show that the solution of formula 2.1.6 is given by:

$$S_{i,t} = S_{i,0}\exp\left(\int_0^t \left(\mu_{i,s} - \frac{1}{2}\sum_{k=1}^{n} v_{ik}^2(s)\right)ds + \int_0^t \sum_{k=1}^{n} v_{ik}(s)dW_{k,s}\right)$$

2.1.3 Basic Financial Terms

A European derivative or contingent claim is a financial contract specified by an exercise time $T \in [0, T^*]$ and a payoff function $H : \{(S_t)\} \times \Omega \to \mathbb{R}$ having primary traded securities, called underlyings, as arguments. The payoff is therefore derived from the underlying, which explains the name. We call it a path-independent European derivative, if we can write $H : \{S_T\} \times \Omega \to \mathbb{R}$, i.e. the payoff is completely determined by the value of the underlying at the exercise time. Otherwise we call it a path-dependent derivative.

If the exercise time is allowed to be a stopping time (with respect to the filtration $\mathbf{F}$), we call it an American derivative.

Definition 2.1.8 Contingent Claim, Derivative Security

A *European contingent claim* or *derivative security* is defined by an expiration date $T \in [0, T^*]$ and any $\mathcal{F}_T$-measurable random variable H representing the payoff.

The most basic kind of a derivative security are futures. The payoff function is defined to be $H(S_T) = S_T - K$. The holder of the future must buy the underlying at time T for the predefined price K. He therefore makes a profit if the underlying is worth more than the strike, otherwise he losses.

Another kind of derivatives are called swaps. In a swap two counterparties agree on an exchange of cash streams. One example is the so called variance swap, for which we will develop a pricing formula at the end of this chapter.

Definition 2.1.9 Variance Swap

A *variance swap* is a financial contract that pays off the difference between the realized, continuously sampled variance $w_T = \frac{1}{T}\int_0^T v_t^2 dt$ over the option's lifetime $[0, T]$ and a prearranged variance which is called the *variance delivery price.*

Our main focus lies on the last example of a derivative security, an option. As the name suggests options are contracts where the holder has the option but not the obligation to execute the derivative. This normally leads to a non-negative payoff function.

The most common options are:

Definition 2.1.10 European K-Call (Put) T Option

The holder of a *European call (put) option* has the right, but not the obligation, to buy (sell) the predefined underlying S at a specific time T, the *exercise time*, for a predefined price K, called the *strike* or *exercise price*.

The payoff of a European option is therefore given by $(S_T - K)^+$ for a call, and $(K - S_T)^+$ for a European put option.
The question to be asked is, how much a fair price for a derivative security would be to pay at time $t \in [0, T]$.

Definition 2.1.11 Price Process
Let us write $\Pi_t(H)$ $(t \in [0, T])$ for a "fair" price of a derivative contract with payoff function H and expiry time T at time t. We call $\Pi_t(H)$ a *price process* for H. The price process is also called the *option's premium* at time t.

The main concept of derivative pricing is the no-arbitrage theory. Loosely speaking, an arbitrage opportunity is a trading strategy with initial cost of 0 and a non-negative payoff in the future. Björk calls this a "deterministic money making machine" ([10], p. 7). Before we can give a mathematical definition we have to introduce some basic financial terms. We start with the term "trading strategy". A trading strategy ϕ defines the amount of units for each single asset hold in a portfolio at time t. We require the process to be predictable. The economic meaning behind this is that traders set their strategy for time t with information up to time t, but not t itself which is mathematically necessary for $\phi \in \mathbf{L}^{d+1}(S)$. Because of assumption i, p. 10, the amount of units $\phi_i(t)$ from asset S_i hold in a portfolio at time t can be every real number.

Definition 2.1.12 Trading (Portfolio) Strategy
Let $T \le T^*$. Any $(d+1)$-dimensional process $\phi = (\phi_0, \ldots, \phi_d)$ with $\phi \in \mathbf{L}(S)$ (where we set $T^* = T$ in EQ 1) is called a *trading strategy* or *portfolio strategy*.

We define the value and gains process for a portfolio strategy by:

Definition 2.1.13 Value and Gains Process
For a trading strategy ϕ we define the *value process* as:

$$V_t(\phi) := \sum_{j=0}^{d} \phi_{j,t} S_{j,t}$$

and the *gains process*:

$$G_t(\phi) := \sum_{j=0}^{d} \int_0^t \phi_{j,s} dS_{j,s}$$

We write $V_t^*(\phi) := \dfrac{V_t(\phi)}{B_t} = \sum_{j=0}^{d} \phi_{j,t} S_{j,t}^*$ for the *discounted value process* and

$G_t^*(\phi) := \sum_{j=0}^{d} \int_0^t \phi_{j,s} dS_{j,s}^*$ for the *discounted gains process*, where S^* is the discounted price process of S.

The value process therefore measures the current worth of the portfolio at time t and the gains process measures the profit or loss up to time t.

Just by crediting our money market account, we could always create a trading strategy with any desired profit at time T. To exclude such strategies we are mainly interested in self-financing strategies. Where a trading strategy with no withdrawals or credits of funds is called a self-financing strategy. Mathematically we define:

Definition 2.1.14 Self-Financing Trading Strategy
A trading strategy ϕ is called *self-financing* if and only if

$$V_t(\phi) = V_0(\phi) + G_t(\phi)$$

In a self-financing strategy changes in the value are only created by movements of the underlying stocks or the money market account: $dV_t(\phi) = dG_t(\phi) = \phi_t dS_t$.
Let us take a look at the discounted value process for a self-financing strategy:

$$\begin{aligned}
dV_t^*(\phi) = d\frac{V_t(\phi)}{B_t} &= V_t(\phi)d\frac{1}{B_t} + \frac{1}{B_t}dV_t(\phi) + d\left\langle V(\phi), \frac{1}{B}\right\rangle_t && \text{(Integration by parts)} \\
&= V_t(\phi)d\frac{1}{B_t} + \frac{1}{B_t}dV_t(\phi) && (B_t \text{ is of finite difference}) \\
&= \phi_t S_t d\frac{1}{B_t} + \frac{\phi_t}{B_t}dS_t && \text{(Self financing strategy)} \\
&= \phi_t dS_t^* && \text{(Discounted underlying)} \\
&= \sum_{j=1}^{d} \phi_{j,t} dS_{j,t}^* && (B_t = S_{0,t})
\end{aligned} \qquad \text{(EQ 4)}$$

From (EQ 4) we see that changes in the discounted value of an self-financing portfolio can only come from changes in the discounted price processes of the assets. This also shows that if $dV_t^*(\phi) = \phi_t dS_t^*$ holds, then ϕ is self-financing.

Because traders set a limit of how much loss they can bear, we can restrain ourselves to only consider portfolio strategies with a lower bounded value process.

Definition 2.1.15 Tame Strategy
A self-financing strategy ϕ is called *tame*, if and only if:

$$\exists K > 0 \forall t \in [0, T] \qquad V_t(\phi) \geq -K$$

Definition 2.1.16 Reachable Claims and Replicating Portfolios
We call a contingent claim H with maturity T *hedgeable* or *reachable* if we can find a self-financing trading strategy ϕ with final value equal to the payoff, i.e. $V_T(\phi) = H$ $\mathbf{P}$-almost surely. Then we call ϕ a *replicating* or *hedging portfolio* for H.[1]

We denote the set of all claims which are reachable by a tame, self-financing strategy with zero initial cost by $\mathcal{K}_0$.

1. Note: as we do not require the discounted value process to be a martingale, this implies that the replicating portfolio may not be unique, see Øksendal [69], p. 274.

We can now define one of the most important properties in financial mathematics: arbitrage opportunity. This definition and the idea of replicating portfolios will be the key in pricing theory.

Definition 2.1.17 Arbitrage Opportunity
A trading strategy ϕ is called an *arbitrage opportunity* for time T, if and only if

$$V_0(\phi) = 0 \quad \wedge \quad V_T(\phi) \geq 0 \quad \wedge \quad \mathbf{P}(V_T(\phi) > 0) > 0$$

almost surely with respect to the probability measure $\mathbf{P}$.

Obviously every market model should be free of arbitrage opportunities:

2.1.4 Arbitrage Free Markets

Definition 2.1.18 (NA) No-Arbitrage
We call a market *arbitrage-free*, if and only if there are no arbitrage opportunities, which are tame.

It is noteworthy that if we could find a measure $\mathbf{Q}$ with the same 0-sets as our real world measure $\mathbf{P}$, and under which every discounted stock is a local martingale, it would follow from (EQ 4) that the discounted value process of a tame strategy ϕ is a supermartingale.[1] This seems to be a desirable property for the following reason:
Let us assume we have an arbitrage opportunity ϕ. Because $\mathbf{Q}$ is equivalent to $\mathbf{P}$ we can also write the properties from an arbitrage opportunity with respect to the new measure $\mathbf{Q}$:

$$\mathbf{Q}(V_0(\phi) = 0) = 1 \qquad \text{(EQ 5)}$$

$$\mathbf{Q}(V_T(\phi) \geq 0) = 1 \qquad \text{(EQ 6)}$$

$$\mathbf{Q}(V_T(\phi) > 0) > 0 \qquad \text{(EQ 7)}$$

From the supermartingale property of the discounted value process we have:

$$\mathbb{E}_Q[V_T^*(\phi)] \leq \mathbb{E}_Q[V_0^*(\phi)] = 0 \qquad \text{(EQ 8)}$$

where we used (EQ 5) in the last equation. We now can use (EQ 8) to get $\mathbb{E}_Q[V_T(\phi)] \leq 0$. From this and (EQ 6) we immediately get $\mathbf{Q}(V_T(\phi) = 0) = 1$, contradicting (EQ 7).

Therefore these kinds of measures prohibit arbitrage opportunities.

Definition 2.1.19 Equivalent Martingale Measure
A probability measure $\mathbf{Q}$ on the filtered probability space $(\Omega, \mathcal{F}, \mathbf{P}, \mathbf{F})$ is called an *equivalent martingale measure*, if and only if:

• $\mathbf{Q}$ is equivalent to $\mathbf{P}$, i.e. $\mathbf{Q}(A) = 0 \Leftrightarrow \mathbf{P}(A) = 0$

1. As an integral with respect to a local martingale, it is a local martingale. Since we assumed ϕ to be tame (lower bounded) it is a supermartingale (application of Fatou's Lemma).

- S^* is a locale martingale with respect to $\mathbf{Q}$.

Definition 2.1.20 (EMM) Existence of an Equivalent Martingale Measure
We say that the market satisfies (*EMM*), if and only if we can find an equivalent martingale measure.

From the above observations we proved the following relationship:

Lemma 2.1.21 Equivalent Martingale Measure Implies an Arbitrage Free Market

$$(\mathrm{EMM}) \Rightarrow (\mathrm{NA})$$

Proof: See above, or Björk [10], pages 137-138. q.e.d.

Note that we only consider tame strategies in the definition of an arbitrage free market. This property, derived from real-world arguments[1], or similar restrictions are necessary from a mathematical point of view, because we can always construct a self-financing, arbitrage opportunity:[2]
Let us write $B_t^* = 1$ for the discounted money market account and let us assume a constant volatility $\nu_t = 1$, so that the discounted underlying is equal to the Brownian motion $S_t^* = W_t$. Let us set the time horizon $T^* = 1$.

We consider the **local** martingale $Y_t = \int_0^t \frac{1}{\sqrt{1-s}} dW_s$ for $t < 1$, with respect to the filtration $(\mathcal{F}_t)$.

The theorem of Dubins/Schwarz[3] implies that Y_t can be written as a stopped Brownian motion $Y_t = \widehat{W}_{\beta(t)}$ with stopping time $\beta(t) = \langle Y \rangle_t = \int_0^t \frac{1}{1-s} ds = \log\left(\frac{1}{1-t}\right)$ and filtration $\mathcal{F}_{\beta(t)}$.

Let us consider the two following stopping times:

$$\tau := \inf(t > 0 \mid \widehat{W}_t = M)$$

and

$$\alpha := \inf(t > 0 \mid Y_t = M)$$

with arbitrary $M > 0$. We know that $\tau < \infty$[4] and $\tau = \log\left(\frac{1}{1-\alpha}\right)$ which implies that $\alpha < 1$ must be satisfied (almost surely).

1. Any market participant has only finite funds available.
2. This example is taken from Øksendal [69], p. 265. It is equivalent to the well known doubling strategy in a time discrete setting, see Etheridge [36], p. 113 or Björk [10], p. 134.
3. See Karatzas, Shreve [60] p. 174.
4. See Karatzas and Shreve [60], p. 96.

Now let us consider the following trading strategy:[1]

$$\phi_t^1 = \begin{cases} \frac{1}{\sqrt{1-t}} & 0 \le t < \alpha \\ 0 & \alpha \le t \le 1 \end{cases}$$

and ϕ_t^0 so that we get a self-financing strategy $\phi_t = (\phi_t^0, \phi_t^1)$ with $V_0(\phi) = 0$.

The discounted value process therefore satisfies $V_t^*(\phi) = \int_0^{t \wedge \alpha} \phi_s^1 dW_s = Y_{t \wedge \alpha}$ and hence we have $V_1^*(\phi) = Y_\alpha = M$ (almost surely) which shows that ϕ is an arbitrage opportunity.

The problem we face here is that the trading strategy ϕ_t is not tame and therefore the discounted value process $V_t^*(\phi)$ is not a supermartingale, but just a local martingale, and the necessary condition $\mathbb{E}_Q[V_T^*(\phi)] \le \mathbb{E}_Q[V_0^*(\phi)]$ used in (EQ 8) is not satisfied.[2]

The obvious question to be asked now is, whether the reverse of lemma 2.1.21 is also true:

Can we find an equivalent martingale measure, if the market is arbitrage free, i.e. $(\mathrm{EMM}) \Leftarrow (\mathrm{NA})$?

It can be shown that in a finite, discrete time setting this is true but as soon as the time index set becomes infinite or we consider infinitely many traded assets, counter examples exist.[3]

We therefore have to strengthen the no-arbitrage condition. In 1981, Kreps [62] introduced the term "No Free Lunch" and showed, that if for every time t the price random variable S_t is bounded, then an equivalent martingale measure exists, if and only if the price process satisfies "No Free Lunch".

Before we can exactly define "No Free Lunch", we need a more functional approach to arbitrage opportunities.

Let us define the Lebesgue space $\mathcal{L}^p = \mathcal{L}^p(\mathbf{P};\mathbb{R}) := \{f : \Omega \to \mathbb{R} \mid f$ is $\mathbf{P}$-measurable and $\|f\|_p < \infty\}$ with $\|f\|_p := \left(\int_\Omega |f|^p d\mathbf{P}\right)^{1/p}$ for $1 \le p < \infty$ and $\|f\|_\infty := \inf_{\mathbf{P}(N) = 0} \sup_{\omega \in \Omega \backslash N} |f(\omega)|$.

Also define:

- $\mathcal{K} := \mathcal{K}_0 \cap \mathcal{L}^\infty$ (see definition 2.1.16 for a definition of $\mathcal{K}_0$) the set of bounded claims, which can be reached by a tame and self-financing portfolio with initial value of 0.

---

1. We have $\phi \in \mathbf{L}(W)$, because $\int_0^1 (\phi_t^1)^2 ds = \int_0^\alpha \frac{1}{1-s} ds = \log\left(\frac{1}{1-\alpha}\right) = \tau < \infty$ almost surely, hence (EQ 1) is satisfied.
2. A more general approach to this problem – in case we want to consider all self-financing strategies in arbitrage situations – is given by the theorem of Dudley [31], see Øksendal [69], p. 267.
3. For an intuitive proof and a counter example in the case of infinitely many assets see Schachermayer [75].

- $\mathcal{L}_+^\infty := \{H \in \mathcal{L}^\infty \mid H \geq 0\}$ all non-negative, bounded random variables.
- $\mathcal{C} := \mathcal{K} - \mathcal{L}_+^\infty$ where we also allow withdrawals $-\mathcal{L}_+^\infty$ from the replicating strategy $\mathcal{K}$.

Lemma 2.1.22 Functional Approach to Arbitrage Free Markets[1]

$$\mathcal{C} \cap \mathcal{L}_+^\infty = \{0\} \Leftrightarrow (\mathrm{NA})$$

Proof: '$\Leftarrow$' Let $H \in \mathcal{C} \cap \mathcal{L}_+^\infty$. This is equivalent to the existence of a tame replicating portfolio ϕ for $H + G$ with $V_0(\phi) = 0$ and $G \in \mathcal{L}_+^\infty$ ($H \in \mathcal{C}$) and $V_T(\phi) = H + G \geq 0$ ($H \in \mathcal{L}_+^\infty$).
If $H \neq 0$ (in $\mathcal{L}^\infty$) we would have $\|H\|_\infty = \inf_{\mathbf{P}(N) = 0} \sup_{\omega \in \Omega \setminus N} |H(\omega)| > 0$, and hence $\mathbf{P}(V_T(\phi) > 0) \geq \mathbf{P}(H > 0) > 0$, where we used $G \geq 0$ in the first inequality. Therefore ϕ is an arbitrage opportunity.
'$\Rightarrow$': On the other hand if we have a tame, self-financing strategy ϕ, with $V_0(\phi) = 0$ and $V_T(\phi) \geq 0$, it follows that $H := V_T(\phi) \in \mathcal{K}$ and $H \in \mathcal{C} \cap \mathcal{L}_+^\infty$.
If $\mathcal{C} \cap \mathcal{L}_+^\infty = \{0\}$ it would follow that $H = 0$, hence $\mathbf{P}(V_T(\phi) > 0) = 0$, i.e. ϕ is not a tame arbitrage opportunity and the market is arbitrage free (as defined in definition 2.1.18). q.e.d.

In our context the following "No Free Lunch" condition is appropriate, as we will see later on:

Definition 2.1.23 (NFLVR) "No Free Lunch with Vanishing Risk"
A financial market model satisfies the *NFLVR* condition if and only if,

$$\bar{\mathcal{C}} \cap \mathcal{L}_+^\infty = \{0\} \quad \text{(EQ 9)}$$

where $\bar{\mathcal{C}}$ denotes the closure of $\mathcal{C}$ in $\mathcal{L}^\infty$, with respect to the norm $\|f\|_\infty$.
For the sake of completeness we will also define the "No Free Lunch" condition:
A financial market model satisfies the *NFL* condition if and only if,

$$\tilde{\mathcal{C}} \cap \mathcal{L}_+^\infty = \{0\} \quad \text{(EQ 10)}$$

where $\tilde{\mathcal{C}}$ denotes the closure of $\mathcal{C}$ in $\mathcal{L}^\infty$, with respect to the weak-star topology of $\mathcal{L}^\infty$.[2]

An explanation and an argumentation why a market should satisfy the "No Free Lunch" condition can be found in Schachermayer [76], page 4. An explanation for NFLVR in Delbaen and Schachermayer [27], page 473, or Björk [10], page 140. Obviously we have the following relationship:[3]

$$(\mathrm{NFL}) \Rightarrow (\mathrm{NFLVR}) \Rightarrow (\mathrm{NA})$$

1. See Björk [10] pp. 138-139.
2. For a definition of the weak-star topology, see Alt [4], p. 215.
3. See Kabanov [58], pp. 18-20.

Theorem 2.1.24 1. Fundamental Theorem of Asset Pricing (Strong)
Let the asset price process be uniformly bounded. Then we have:
A financial market model has got an equivalent martingale measure, if and only if the market satisfies the "no free lunch with vanishing risk" condition.

$$(\mathrm{EMM}) \Leftrightarrow (\mathrm{NFLVR})$$

Proof:[1] We will assume that the asset price process is bounded. By means of standard arguments we can prove the general result for uniformly bounded processes.

"$\mathrm{EMM} \Rightarrow \mathrm{NFLVR}$" The easier part can be proven without any boundedness' conditions on the price processes. Let $\mathbf{Q}$ denote the equivalent martingale measure and g the Radon-Nikodym derivative, i.e. $g d\mathbf{P} = d\mathbf{Q}$. Since the discounted value process of the replicating strategy belonging to $H \in \mathcal{K}$ is a $\mathbf{Q}$-supermartingale with initial costs of zero, we have:

$$\mathbb{E}_Q[H] = \int H(\omega) g(\omega) d\mathbf{P}(\omega) \leq 0$$

and hence for all $H \in \mathcal{C}$:

$$\mathbb{E}_Q[H] = \int H(\omega) g(\omega) d\mathbf{P}(\omega) \leq 0 \qquad \text{(EQ 11)}$$

Because $g \in \mathcal{L}^1$, we have that g is weak-star continuous. As a consequence (EQ 11) is valid for all $H \in \tilde{\mathcal{C}}$. If now we assume $H \in \tilde{\mathcal{C}} \cap \mathcal{L}^{\infty}_{+}$ we immediately retrieve $H = 0$, which proves (NFL) and therefore also the weaker condition (NFLVR).[2]

"$\mathrm{NFLVR} \Rightarrow \mathrm{EMM}$" The main idea of the proof is to use a separation theorem on the two convex sets $\mathcal{C}$ and $\mathcal{L}^{\infty}_{+}$ in $\mathcal{L}^{\infty}$, which would give us a positive, random variable $X \in \mathcal{L}^1$ with

$$\forall Y \in \mathcal{C} \quad \mathbb{E}[XY] \leq 0$$

Because $X > 0$ we can scale X, so that we have $\mathbb{E}[X] = 1$, therefore we can use X as a Radon-Nikodym derivative to retrieve a new measure $\forall A \in \mathcal{F}_T \; \mathbf{Q} := \int_A X d\mathbf{P}$. Hence we can write the expectation with respect to the new measure as:

$$\mathbb{E}_Q[Y] = \mathbb{E}[XY]$$

We will now show that $\mathbf{Q}$ is an equivalent martingale measure.

1. See Björk [10], p. 141.
2. See Schachermayer [76], p. 8.

Because $\mathbb{E}[XY] \le 0$ is satisfied for all $Y \in \mathcal{K}$ and with $Y \in \mathcal{K}$ we have $-Y \in \mathcal{K}$, we get:

$$\forall Y \in \mathcal{K} \qquad \mathbb{E}[XY] = 0 \tag{EQ 12}$$

To prove the martingale property of the discounted stocks, we will consider the following tame self-financing strategy $\phi = I_A(s)$ with $s \le t$ and $A \in \mathcal{F}_s$ fixed:

- Let $V_0(\phi) = 0$ and do nothing till time s
- At time s buy $\mathbf{I}\{A\}$ units of asset number i and finance this by a loan in the bank.
- At time t sell the assets, pay the proceeds of the sale into the bank account and keep it there till time T.

(where $\mathbf{I}\{.\}$ denotes the characteristic function.)

Hence this portfolio strategy has the discounted value of $V^*_T(\phi) = V^*_t(\phi) = \mathbf{I}\{A\}[S^*_i(t) - S^*_i(s)]$.

We therefore have $\mathbf{I}\{A\}[S^*_i(t) - S^*_i(s)] \in \mathcal{K}$. Using (EQ 12) shows $\mathbb{E}_Q[\mathbf{I}\{A\}[S^*_i(t) - S^*_i(s)]] = 0$ and because this is valid for all $s \le t$ and $A \in \mathcal{F}_s$ we proved that S^*_i is a $\mathbf{Q}$ Martingale. q.e.d.

The challenge is to retrieve a separation theorem, which yields X with the desired properties. Unfortunately standard separation theorems like Hahn-Banach Space Separation Theorem or Convex Separation Theorem cannot be applied.[1] We therefore need a stronger one:

Theorem 2.1.25 Kreps-Yan, Schaefer Separation Theorem

If $\mathcal{C}$ is weak-star closed, and if $\mathcal{C} \cap \mathcal{L}^\infty_+ = \{0\}$, then we can find $X \in \mathcal{L}^1$ with $X > 0$ and

$$\forall Y \in \mathcal{C} \ \mathbb{E}[XY] \le 0$$

Proof: It mainly follows from Schaefer [78], see Schachermayer [76] pp. 8-9. q.e.d.

It follows that we can apply theorem 2.1.25 in theorem 2.1.24 and the prove given there is completed, if we can show that $\mathcal{C}$ is weak-star closed. This is the difficult part of the proof, it is a result from Delbaen and Schachermayer [27] derived in 1994:

Lemma 2.1.26 NFLVR Implies Weak-Star Closed

Let the asset price process be uniformly bounded then we have:

$$(\text{NFLVR}) \quad \Rightarrow \quad \mathcal{C} \text{ weak-star closed}$$

Proof: Delbaen and Schachermayer [27]. q.e.d.

Remark. The assumption that the asset price process is uniformly bounded is valid for our market model, because we only consider continuous processes. But it is also valid for a much broader spectrum, for example all processes with bounded jumps.

1. See Björk [10] p. 139.

The theorem only gives us an equivalent measure under which each discounted asset price process is a local martingale. To retrieve a (real) martingale we would have to strengthen the assumption to only consider bounded processes, as can be seen in the proof.

With the help of Girsanov's theorem[1] we can try to find an equivalent martingale measure. Let us assume that we have a new probability measure $d\mathbf{Q} = \mathcal{E}(X \bullet W)_t d\mathbf{P}$ with a new Brownian motion $dW_t^* := dW_t + X_t dt$. The discounted underlying process satisfies:

$$\forall i \in \{1, \ldots, d\} \qquad dS_{i,t}^* = S_{i,t}^*(\mu_{i,t} - r)dt + S_{i,t}^* \sum_{k=1}^{n} \nu_{ik}(t) dW_{k,t}$$

which can be written with respect to the new Brownian Motion W_t^* as

$$dS_{i,t}^* = S_{i,t}^*\left(\mu_{i,t} - r - \sum_{k=1}^{n} X_{k,t} \cdot \nu_{ik}(t)\right)dt + S_{i,t}^* \sum_{k=1}^{n} \nu_{ik}(t) dW_{k,t}^* \qquad \text{(EQ 13)}$$

If the discounted stock process (EQ 13) should be a local martingale (with respect to $\mathbf{Q}$) the drift has to be 0, which implies that the following equation must have a solution ($\mathbf{P}$-almost surely), for every $i \in \{1, \ldots, d\}$ and every $t \le T$:

$$\mu_{i,t} - r - \sum_{k=1}^{n} X_{k,t} \cdot \nu_{ik}(t) = 0$$

Definition 2.1.27 Market Price of Risk Process
Every n-dimensional stochastic process $X = (X_1, \ldots, X_n) \in \mathbf{L}(W)$ so that the Dolean exponential $\mathcal{E}(X \bullet W)$ is a martingale, e.g. Novikov's condition is satisfied, is called a *market price of risk process* if it solves the linear system of equations given by

$$\nu X = \mu - r1_d \qquad \text{(EQ 14)}$$

where 1_d is the d-dimensional vector only consisting of one's, i.e. $1_d = (1, \ldots, 1)$, $\mu = (\mu_1, \ldots, \mu_d)$ the drift and ν the $d \times n$ volatility matrix.

Ignoring integrability problems. We notice – provided that the volatility matrix $\nu(t) : \mathbb{R}^n \to \mathbb{R}^d$ is surjective for each $t \le T$ – we can find an equivalent martingale measure (EQ 14) and the market model is arbitrage free (lemma 2.1.21). The matrix can only be surjective if it has got rank d which implies $d \le n$, i.e. the market model must have at least as many random sources as it has tradeable assets.
The exact relationship between the existence of arbitrage opportunities and solutions to (EQ 14) is expressed in the following theorem.

Theorem 2.1.28 1. Fundamental Theorem of Asset Pricing (Weak)
If we can find a market price of risk process the market is arbitrage free. Conversely if the market is arbitrage free then (EQ 14) has a solution $X \in \mathbf{L}^n(W)$ (but $\mathcal{E}(X \bullet W)$ is not necessarily a martingale).

1. See appendix A.2.

Proof: See Øksendal [69], pages 268-270. q.e.d.

Static Arbitrage. One real-life problem with the definition of an arbitrage opportunity is that it depends on the information set (filtration $\mathbf{F} = (\mathcal{F}_t)$). To establish that a given set of prices are arbitrage free, one must know the possible future price paths which in real life is impossible. One way to avoid this is to restrict the information set on which trading strategies can rely on. For example Carr et al. [18] introduced the term "static arbitrage" and Carr and Madan [19] derived that if a set of option price quotes is free of bull-, butterfly- and calendar spread arbitrages it is free of all static arbitrage opportunities.

Because we will come across a butterfly spread once again, we will define it here. For a definition of the other spreads see Carr and Madan [19], page 127-129.
Let $K_{i-1} < K_i < K_{i+1}$ be three different strike prices and let $C(K, T)$ denote a long call option with strike K and time to expiry T. We define a butterfly strategy by:

$$BS_i(T) = C(K_{i-1}, T) + \left(\frac{K_i - K_{i-1}}{K_{i+1} - K_i}\right) C(K_{i+1}, T) - \left(\frac{K_{i+1} - K_{i-1}}{K_{i+1} - K_i}\right) C(K_i, T)$$

Because the butterfly strategy pays off a non-negative amount, we call it a butterfly arbitrage if we can find strike prices $K_{i-1} < K_i < K_{i+1}$ and time to expiry of T with $BS_i(T) < 0$.

2.1.5 Complete Markets

Let us assume that the volatility matrix $\nu(t) : \mathbb{R}^n \to \mathbb{R}^d$ is injective for all $t \le T$ which implies $n \le d$, i.e. the market model must have at least as many tradeable assets as random sources. As a consequence (EQ 14) has got no solution or the solution is unique. Can we deduce that the equivalent martingale measure would be unique as well?
In general, where randomness is not only induced by Brownian motion, this is not true, because under these assumptions not every equivalent measure is of the Girsanov type. However, in our case we will prove that every equivalent measure is of the Girsanov type.

Definition 2.1.29 Complete Market
We call a financial market model *complete* if and only if every derivative security H with $\mathbb{E}_Q[|H|/S_{0,T}] < \infty$ is attainable. Otherwise we call the market *incomplete*.

Amazingly a strong connection exists between the uniqueness of the martingale measure and the amount of $\mathbf{Q}$-attainable derivatives, which we will discern in the following.

Theorem 2.1.30 Inverse Girsanov
Let $W_t = (W_{1,t}, \ldots, W_{n,t})$ be an n-dimensional $\mathbf{P}$-Brownian motion and let $\mathbf{F} = (\mathcal{F}_t)$ be a filtration generated by the Brownian motion W_t.

If we have another probability measure $\mathbf{Q}$ which is equivalent to $\mathbf{P}$ on $\mathcal{F}_T$, then $\mathbf{Q}$ is coming from a Girsanov transformation, i.e. there exists an adapted process X_t, so that the process $L_t = \frac{d\mathbf{Q}}{d\mathbf{P}}$ on $\mathcal{F}_t$ derived from the Radon-Nikodym theorem (called the likelihood process) has the following dynamics:

$$dL_t = -L_t X_t dW_t \quad \wedge \quad L_0 = 1 \qquad \text{(EQ 15)}$$

Proof:[1] The likelihood process is a $\mathbf{P}$-martingale. From the martingale representation theorem[2] we know that an n-dimensional, $\mathbf{F}$-adapted process $Y_t = (Y_{1,t}, \ldots, Y_{n,t})$ exists which satisfies:

$$L_t = L_0 + \sum_{k=1}^{n} \int_0^t Y_{k,s} dW_{k,t}$$

Now set $X_t := -Y_t / L_t$. q.e.d.

Lemma 2.1.31 Functional Approach to Complete Markets[3]

Let us assume that we have an equivalent martingale measure $\mathbf{Q}$ and that randomness is only induced by Brownian motion. We then have:

$$\mathbf{Im}(\hat{v}(t)) = \mathbb{R}^n \quad \Leftrightarrow \quad \text{model is complete}$$

where $\hat{v}(t)$ denotes the transpose of the matrix $v(t)$.

Proof: Let us assume we have a contingent claim with payoff H and $\mathbb{E}_Q[|H| / S_{0,T}] < \infty$. Considering the following martingale

$$M_t := \mathbb{E}_Q\left[\frac{H}{S_{0,T}} \mid \mathcal{F}_t\right] \qquad \text{(EQ 16)}$$

we know that H can be hedged if and only if we can write:[4]

$$dM_t = h_t dS_t^* \qquad \text{(EQ 17)}$$

Because $\mathcal{F}_t$ is a filtration generated from W_t and not from W_t^* we cannot apply the martingale representation theorem directly to (EQ 16) to get $dM_t = g_t dW_t^*$.[5]

1. See Björk [10], p. 164. However, Björk only assumes that $\mathbf{Q}$ is absolute continuous to $\mathbf{P}$. This leads to the problem that the likelihood process could be zero. We avoided this difficulty by assuming that the measures are equivalent.
2. See appendix A.3.
3. See Björk [10], p. 197.
4. See Björk [10], pp. 145-146.
5. See Karatzas and Shreve [60], p. 375.

Applying the Bayes rule on (EQ 16) yields:

$$M_t = \frac{1}{L_t}\mathbb{E}\left[L_T\frac{H}{S_{0,T}} \mid \mathcal{F}_t\right] \qquad \text{(EQ 18)}$$

where L_t is the likelihood process.

Now we can apply the martingale representation theorem to the $\mathbf{P}$-martingale $N_t := \mathbb{E}\left[L_T\frac{H}{S_{0,T}} \mid \mathcal{F}_t\right]$ and get Y_t with:

$$dN_t = Y_t dW_t$$

Applying Itô's formula to (EQ 15) in theorem 2.1.30[1] yields $d\frac{1}{L_t} = \frac{X_t}{L_t}dW_t + \frac{X_t^2}{L_t}dt$.

Using integration by parts to $M_t = \frac{1}{L_t} \cdot N_t$ yields:

$$\begin{aligned} dM_t &= \frac{1}{L_t}dN_t + N_t d\frac{1}{L_t} + d<\frac{1}{L_t}, N_t> \\ &= \frac{Y_t}{L_t}dW_t + N_t\frac{X_t}{L_t}dW_t + N_t\frac{X_t^2}{L_t}dt + Y_t\frac{X_t}{L_t}dt \\ &= \frac{Y_t - N_tX_t}{L_t}W_t^* \\ &= g_t dW_t^* \end{aligned} \qquad \text{(EQ 19)}$$

with $g_t = \frac{Y_t - N_tX_t}{L_t}$. Where we used $dW_t^* = dW_t + X_t dt$ for the third equality.

On the other hand we know that $dS_t^* = S_t^*\nu(t)dW_t^*$. Inserting this into (EQ 17) and comparing coefficients between (EQ 17) and (EQ 19) we derive $g_t = h_t S_t^* \nu(t)$ with

$$S_t^* = \begin{bmatrix} S_{1,t}^* & 0 & 0 \\ 0 & \dots & 0 \\ 0 & 0 & S_{0,t}^* \end{bmatrix}$$

hence it is equivalent to $\hat{g}_t = \hat{\nu}(t)S_t^*\hat{h}_t$ This equation can be solved if and only if we have $\hat{g}_t \in \mathbf{Im}(\hat{\nu}(t))$. q.e.d.

1. Note that we assumed that randomness is only induced by Brownian motion.

Theorem 2.1.32 2. Fundamental Theorem of Asset Pricing

Assuming we have an equivalent martingale measure $\mathbf{Q}$:
The martingale measure is unique if and only if the market model is complete.

Proof:[1] Within our model assumptions, it immediately follows from lemma 2.1.31 because we have that the orthogonal complement of the image of the transposed volatility matrix is equal to the kernel of the volatility matrix:

$$\{\mathbf{Im}(\hat{\nu}(t))\}^{\perp} = \mathbf{Ker}(\nu(t))$$

we see that the market model is complete if and only if we have:

$$\mathbf{Ker}(\nu(t)) = \{0\}$$

which is equivalent with the uniqueness of the equivalent martingale measure, because every equivalent measure is of the Girsanov type (theorem 2.1.30).
Note however that this theorem is true in much more general models, see for example Björk [10], pages 146-147, or Harrison and Pliska [53]. q.e.d.

The completeness of a financial market arises the question why options are traded anyway, if the same payoff structure can be achieved with a trading strategy on the underlyings? The question is comprehensively answered in Cox and Rubinstein [24], pages 44-59. We just want to point out the fact that the market is not frictionless and that traders cannot trade continuously (assumption i). Especially during market turmoil or crashes it may and was very difficult for traders to get rid of unwanted positions, e.g. during market turmoil in 1987. Therefore even though theoretically every contingent claim is attainable, this may be impossible or – due to transaction costs – too expansive in practice.

2.1.6 Martingale Pricing

Theorem 2.1.33 No Arbitrage Asset Pricing

Let us assume we have an arbitrage free market, and a replicating portfolio ϕ for the contingent claim H with maturity T. We then have:

$$\forall t \in [0, T] \qquad \Pi_t(H) = V_t(\phi)$$

Additionally, if H can be hedged by ϕ and ψ, we have:

$$\forall t \in [0, T] \qquad V_t(\phi) = V_t(\psi) .$$

Proof:[2] The equations follow immediately from definition 2.1.18 and definition 2.1.16. q.e.d.

1. See Björk [10], p. 198.
2. See Björk [10], p. 112.

Theorem 2.1.34 Risk-Neutral Asset Pricing[1]

Let $\mathbf{Q}$ be an equivalent martingale measure and H be any contingent claim with maturity T. We have:

$$\forall t \in [0, T] \qquad \Pi_t(H) = S_{0,t} \mathbb{E}_Q\left[\frac{H}{S_{0,T}} \mid \mathcal{F}_t\right].$$

Proof: This can be proven by using the martingale property of the discounted price process.q.e.d.

(NFLVR) We therefore have a way to get a no-arbitrage price $\Pi_0(H)$ for every derivative. If the equivalent martingale measure should be unique, so would be the risk neutral price derived in theorem 2.1.34. If the contingent claim should be attainable we have two independent ways to retrieve a pricing process. However, simple calculations show that the price process of theorem 2.1.33 coincided with the one from theorem 2.1.34.[1] And in this case the price process will be unique and independent of the chosen martingale measure or the hedging portfolio. As a consequence we have in a complete market that every contingent claim can be uniquely priced with theorem 2.1.33 or theorem 2.1.34.

(NA) On the other hand if the market is arbitrage-free and complete every derivative can be uniquely priced with theorem 2.1.33, but because we do not necessarily have an equivalent martingale measure, we cannot apply theorem 2.1.34.

2.1.7 Stock Price Distribution

Deterministic Volatility

If we assume that the volatility matrix $\nu_{i,k}(t)$ and the drift vector $\mu_{i,t}$ are deterministic we can see that the stock price processes $S_{i,t}$ are lognormal distributed:

Lemma 2.1.35 Deterministic Volatility Implies Lognormal Distributed Stocks

Let S_t be given by:

$$S_t = S_0 \exp\left(\int_0^t \left(\mu_s - \frac{1}{2}\sum_{k=1}^n \nu_k^2(s)\right) ds + \sum_{k=1}^n \int_0^t \nu_k(s) dW_{k,s}\right).$$

If μ and ν_k for $k \in \{1, \ldots, n\}$ are deterministic functions and W_t is a standard n-dimensional Brownian motion, then S_t is lognormal distributed, i.e.

$$\mathbf{P}\left(\log\left(\frac{S_t}{S_0}\right) < a\right) = \mathbf{N}(a)$$

1. See Björk [10], pp. 148.

where **N** is the normal distribution with mean:

$$m = \int_0^t \left(\mu_s - \frac{1}{2}\sum_{k=1}^{n} \nu_k^2(s)\right) ds$$

and variance:

$$v = \sum_{k=1}^{n} \int_o^t \nu_k(s)^2 ds$$

Proof: Since we assumed a deterministic drift and volatility process we have

$$\log\left(\frac{S_t}{S_0}\right) = m + \int_0^t \sum_{k=1}^{n} \nu_k(s) dW_{k,s}$$

with $m \in \mathbb{R}$ a constant. Obviously we are finished if we can show that $X_t := \sum_{k=1}^{n} \int_0^t \nu_k(s) dW_{k,s}$ is a normal distributed random variable with mean 0 and variance v. Consider

$$X_{k,t} := \int_0^t \nu_k(s) dW_{k,s}$$

we can approximate this integral with Riemann sums

$$R_n := \sum_{i=0}^{n} \nu_k(t_i)[W_k(t_{i+1}) - W_k(t_i)] \ ,$$

where $\Pi = \{t_0, \ldots, t_n\}$ is a partition of $[0, t]$ with $0 = t_0 < \ldots < t_n = t$.

Obviously R_n is normal distributed with mean zero and variance $v_k := \sum_{i=0}^{n} \nu_k(t_i)^2 (t_{i+1} - t_i)$.

We have $R_n \to X_{k,t}$ almost surely and $v_k \to \int_o^t \nu_k(s)^2 ds$ if $\max(|t_{i+1} - t_i|) \to 0$. Hence we have that R_n is normal distributed with mean zero and variance $\int_o^t \nu_k(s)^2 ds$. Which completes the proof. However the last argument would have to be made precise. q.e.d.

General Case

Theorem 2.1.34 gives us the formula to price a call option with strike K and expiry T:

$$\begin{aligned} C(K,T) &= \exp(-rT)\mathbb{E}_Q[(S_T-K)^+] \\ &= \exp(-rT)\int_{x=K}^{\infty}(x-K)g(x)dx \end{aligned} \qquad \text{(EQ 20)}$$

with $g(x)$ the risk-neutral density (RND) defined by $\mathbf{Q}(S_T<y) := \int_0^y g(x)dx$.

As was shown by Breeden and Litzenberger [16], we can differentiate (EQ 20) twice with respect to the strike K to derive the risk-neutral density $g(x)$ of the stock price at expiry:

$$g(x) = \exp(rT)\frac{\partial^2}{\partial K^2}C(K,T)\bigg|_{K=x} \quad . \qquad \text{(EQ 21)}$$

We can approximate this with a butterfly spread:[1]

$$g(x) = \exp(rT)\frac{C(K-\varepsilon,T)+C(K+\varepsilon,T)-2C(K,T)}{\varepsilon^2} \qquad \text{(EQ 22)}$$

This formula will later be applied to analyse the relationship between the RND and the volatility of volatility as well as between the RND and the correlation between the volatility and the stock price process (see chapter 2.4.2).

2.1.8 Why Use a Geometric Brownian Motion For Stock Modelling?

So far we did not give any explanation for assumption iii, i.e. the way of how we model the stock movements. We will make it good now.
Already in the late 1950's stock price movements were modelled under the assumption that they are lognormal distributed. For example, Osborne [70] came to the conclusion that the log-returns of the stock $\log(S_T/S_0)$ are normal distributed with mean 0 and variance $\nu\sqrt{T}$.

Let us start with the assumption, that the price processes are continuous. This assumption makes it easier from a mathematical point of view, even though the basic techniques can be generalized for a non-continuous setup.[2]

As we already have seen the assumption of an (essential) arbitrage free market leads to the existence of an equivalent martingale measure $\mathbf{Q}$; by definition $M_t := S_t^*$ is a local martingale with respect to

1. See page 21 for a definition of a butterfly spread. See Hull [55], p. 389 for a proof. See Carr and Madan [19], p. 128 for the financial background.
2. For example Itô's formula, see Etheridge [36], p. 176.

$\mathbf{Q}$. If we now assume that the discounted stock movements are not very rough, i.e. having a finite p-variation[1] ${}^p\langle M\rangle_T < \infty$ for any $p < 2$, we deduce that $M_t = S_0^*$ is satisfied almost surely.[2]

Consequently we would have $S_t = S_0e^{rt}$ which of course does not reflect real stock prices.

We therefore see that the increments of the discounted stock price process M_t must be a $\mathbf{Q}$-local martingale and ${}^p\langle M\rangle_T = \infty$ for $p < 2$.[3]

We could now assume that the increments have stationary conditional variance, but since we assumed that M_t is continuous this would imply that M_t would be a Brownian motion (Levy's characterisation), and it therefore has to be neglected.
On the other hand if the relative increments $(M_t - M_s)/M_s$ have stationary conditional variance, $\log(M_t)$ would have stationary increments. So let us assume:

$$\langle\log(M)\rangle_t = v^2t \qquad \text{(EQ 23)}$$

Because we can apply Itô's formula to $\log(x)$ we have:

$$d\langle\log(M)\rangle_t = \frac{1}{M_s^2}d\langle M\rangle_s$$

And hence, using (EQ 23), yields:

$$d\langle M\rangle_t = v^2M_s^2ds$$

Which gives us one SDE for the discounted stock price process:

$$dS_t^* = dM_t = vM_tdW_t^*$$

and one for the price process:

$$dS_t = rS_tdt + vS_tdW_t^*$$

Because we know that $\mathbf{Q}$ must come from a Girsanov transformation we have $dW_t^* = dW_t + X_tdt$, and as a consequence:

$$dS_t = \mu S_tdt + vS_tdW_t$$

with $\mu = r + vX_t$. This is the financial market model first derived by Samuelson [74].[4]

1. Where the p-variation is defined as ${}^p\langle M\rangle_T := \lim_{\delta\to 0}\left\{\sup(\sum_{j=1}^{N(\pi)}|M_j - M_{j-1}|^p)\right\}$, where we set $M_j = M_{t_j}$ and the supremum is taken over all partitions $\pi = (t_0, \ldots, t_{N(\pi)}) \subseteq [0, T]$ with $t_0 = 0 \wedge t_{N(\pi)} = T$ and $\max(|t_j - t_k| \mid j, k \in \{0, N(\pi)\}) = \delta$, i.e. having a mesh of δ.
2. Every continuous, local martingale with a finite p-variation ($p < 2$) is constant.
3. See Etheridge [36], pp. 72-74, and the references cited there.
4. See also Etheridge [36], p. 102.

2.1.9 Applications of Chapter 2.1.6, “Martingale Pricing“

Lemma 2.1.36 Put-Call Parity

Let $C_t(K, T)$ be the price process of a European call and $P_t(K, T)$ the one of a European put option with strike price K and maturity T on the stock S_t. Following connection exists:

$$C_t(K, T) - P_t(K, T) = S_t - \frac{B_t}{B_T}K$$

Proof: Let us consider the payoff functions $(S_T - K)^+ - (K - S_T)^+ = S_T - K$. At maturity the value of a long call and a short put is equal to the value of a long forward. A forward can easily be replicated by a self-financing portfolio $\phi = (-K/B_T, 1)$. From theorem 2.1.33 we know that the price process of a long call and a short put must be the same as the value process of the replicating portfolio $V_t(\phi) = (-K/B_T) \cdot B_t + 1 \cdot S_t$. q.e.d.

Lemma 2.1.37 Replicating Smooth Contingent Claims with Standard Options

If we assume that we have European call and put options with maturity T and strike ranging from 0 to infinity, we can replicate every twice differentiable payoff function f with standard put and call options.

Formula 2.1.38 Twice differentiable payoff replicating with standard options

$$f(S) = f(\kappa) + f'(\kappa)(S - \kappa) + \int_{\kappa}^{\infty} f''(v)(S - v)^+ dv + \int_{\kappa}^{\infty} f''(v)(v - S)^+ dv$$

Proof:[1] The fundamental theorem of calculus implies for any arbitrary $\kappa \in \mathbb{R}$:

$$f(S) = f(\kappa) + \mathbf{1}\{S > \kappa\}\int_{\kappa}^{S} f'(u)du - \mathbf{1}\{S < \kappa\}\int_{S}^{\kappa} f'(u)du$$

$$= f(\kappa) + \mathbf{1}\{S > \kappa\}\int_{\kappa}^{S}\left[f'(\kappa) + \int_{\kappa}^{u} f''(v)dv\right]du - \mathbf{1}\{S < \kappa\}\int_{S}^{\kappa}\left[f'(\kappa) - \int_{u}^{\kappa} f''(v)dv\right]du$$

With Fubini‘s theorem (changing the order of integration) we get:

$$f(S) = f(\kappa) + f'(\kappa)(S - \kappa) + \mathbf{1}\{S > \kappa\}\int_{\kappa}^{S}\int_{v}^{S} f''(v)dudv + \mathbf{1}\{S < \kappa\}\int_{\kappa}^{S}\int_{v}^{S} f''(v)dudv$$

1. See Carr and Madan [21], pp. 417-428.

Integrating over u yields the desired result:

$$f(S) = f(\kappa) + f'(\kappa)(S-\kappa) + \mathbf{I}\{S>\kappa\}\int_{\kappa}^{S} f''(v)(S-v)dv$$

$$+\mathbf{I}\{S>\kappa\}\int_{S}^{\kappa} f''(v)(v-S)dv$$

$$= f(\kappa) + f'(\kappa)(S-\kappa) + \int_{\kappa}^{\infty} f''(v)(S-v)^{+}dv + \int_{\kappa}^{\infty} f''(v)(v-S)^{+}dv$$

q.e.d.

Lemma 2.1.39 Pricing of Variance Swaps

Assumption i) There are European call and put options with maturity T and strike ranging from 0 to infinity.

Assumption ii) The underlying stock price process S_t is defined as in assumption iii, p. 10:

$$\frac{dS_t}{S_t} = \mu_t dt + \nu_t dW_t \qquad \text{(EQ 24)}$$

If we assume NFLVR we can replicate a T-maturity variance swap (definition 2.1.9) and its theoretical value at time $t = 0$ is given by:

Formula 2.1.40 Pricing of a Variance Swap

$$K_{VARS} = \mathbf{IE}_Q[w_T] = \frac{2}{T}e^{rT}\left(\int_0^{F_0(T)} \frac{1}{K^2}P(K,T)\,dK + \int_{F_0(T)}^{\infty} \frac{1}{K^2}C(K,T)\,dK\right)$$

where $F_0(T)$ is the future value of the stock price, i.e. $F_0(T) = S_0 e^{rT}$, $P(K,T)$ and $C(K,T)$ the price of a European put and call option, respectively, with strike K and maturity T

Proof:[1] Applying Itô's lemma to (EQ 24) we get

$$d\log S_t = \left(\mu_t - \frac{1}{2}\nu_t^2\right)dt + \nu_t dW_t \qquad \text{(EQ 25)}$$

1. For a similar proof see Hafner [51], pp.201-204.

by subtracting (EQ 24) from (EQ 25) and integrating from 0 to T yields:

$$\int_0^T \frac{dS_t}{S_t} - \log\left(\frac{S_T}{S_0}\right) = \frac{1}{2}\int_0^T \nu_t^2 dt$$

if we now divide by T we get a formula for the payoff from a variance swap, i.e. the realized variance w_T as defined in definition 2.1.9:

$$w_T = \frac{2}{T}\left[\int_0^T \frac{dS_t}{S_t} - \log\left(\frac{S_T}{S_0}\right)\right]$$

Where $\int_0^T \frac{dS_t}{S_t}$ represents the payoff from a trading strategy which involves continuous rebalancing of a stock position and $-\log(S_T/S_0)$ represents the payoff from a static short position in a log-contract. Because of NFLVR (theorem 2.1.34) we have that the price for a variance swap K_{VARS} (at time 0) is given by $\exp(-rT)\mathbb{E}_Q[w_T]$, we hence can write:

$$K_{VARS} = \mathbb{E}_Q[w_T] = \frac{2}{T}\cdot\mathbb{E}_Q\left[\int_0^T \frac{dS_t}{S_t}\right] - \frac{2}{T}\cdot\mathbb{E}_Q\left[\log\left(\frac{S_T}{S_0}\right)\right] \qquad \text{(EQ 26)}$$

We can easily calculate the first expectation:

$$\mathbb{E}_Q\left[\int_0^T \frac{dS_t}{S_t}\right] = \mathbb{E}_Q\left[\int_0^T r dt + \int_0^T \nu_t dW_t^*\right] = rT \qquad \text{(EQ 27)}$$

Because the log-contract is not a market-traded security we have to replicate it with traded options. For that we try to rewrite the payoff from the log-contract $\log(a/b)$ with the payoff from European standard put and call options. If we set $f(S) = \log(S_T)$ and $\kappa = S_*$ in lemma 2.1.37 we get

$$\log(S_T) = \log(S_*) + \frac{S_T - S_*}{S_*} - \int_{S_*}^{\infty} \frac{(S_T - K)^+}{K^2} dK - \int_0^{S_*} \frac{(K - S_T)^+}{K^2} dK$$

where S_* is an arbitrary stock price with

$$\log\left(\frac{S_T}{S_0}\right) = \log\left(\frac{S_T}{S_*}\right) + \log\left(\frac{S_*}{S_0}\right) \qquad \text{(EQ 28)}$$

so that the second term on the right side is constant and known at time $t = 0$.

We therefore have replicated a log-contract with standard options.

Taking expectations and making use of Fubini, we get:

$$\mathbb{E}_Q\left[\log\left(\frac{S_T}{S_*}\right)\right] = \frac{\mathbb{E}_Q[S_T]-S_*}{S_*} - \mathbb{E}_Q\left[\int_{S_*}^{\infty}\frac{(S_T-K)^+}{K^2}dK - \int_0^{S_*}\frac{(K-S_T)^+}{K^2}dK\right]$$
$$= \frac{(S_0e^{rT}-S_*)}{S_*} - e^{rT}\int_{S_*}^{\infty}\frac{C(K,T)}{K^2}dK - e^{rT}\int_0^{S_*}\frac{P(K,T)}{K^2}dK \qquad \text{(EQ 29)}$$

Inserting (EQ 27), (EQ 28) and (EQ 29) into (EQ 26) yields

$$K_{VARS} = \frac{2}{T}\left[rT - \log\left(\frac{S_*}{S_0}\right) - \frac{(S_0e^{rT}-S_*)}{S_*} + e^{rT}\int_{S_*}^{\infty}\frac{C(K,T)}{K^2}dK + e^{rT}\int_0^{S_*}\frac{P(K,T)}{K^2}dK\right] \qquad \text{(EQ 30)}$$

which proves formula 2.1.40, if we set $S_* = S_0e^{rT}$. q.e.d.

2.1.10 Summary

Putting everything together and ignoring integrability problems, we saw that the following connection exists between the number of random sources n and the number of tradeable assets d (excluding the bank account):[1]

- $d \le n$. The volatility matrix $(\nu_{ij}(t))$ is surjective. We can find an equivalent martingale measure (Girsanov theorem) which implies that the market is arbitrage free (lemma 2.1.21).
- $d \ge n$. The volatility matrix $(\nu_{ij}(t))$ is injective. We saw that the market is complete (Inverse Girsanov, theorem 2.1.30) which implies that there is at maximum one martingale measure (theorem 2.1.32).
- $d = n$. The volatility matrix $(\nu_{ij}(t))$ is bijective. We therefore have a complete and arbitrage free market model with exactly one martingale measure.

We proved following Björk [10] and Schachermayer [76] that in the context of an (essential) arbitrage free market we get a no-arbitrage price for every derivative, depending on the chosen equivalent martingale measure. If the derivative would be attainable, this price is unique. Hence in a complete and arbitrage-free market every contingent claim can be uniquely priced. At the end of this chapter we applied this theorem to get a pricing formula for a variance swap, the proof is similar to the one found in Hafner [51].

1. Björk [10] calls this a “meta-theorem”, p. 118.

CHAPTER 2.2 The Black-Scholes Model

The Black-Scholes model is the most widely known and used model for modelling financial instruments. It can be used to retrieve a closed formula for pricing European put and call options. This equation was derived by Fischer Black (1938 - 1995) and Myron Scholes (born 1941) in a paper published in 1973.[1] Their work and this of Robert Carhart Merton (born 1944) was honoured by receiving the Bank of Sweden Prize in Economic Sciences in Memory of Alfred Nobel in the year 1997.

The advantage of this model is that we will be able to derive a closed formula for pricing European call and put options and that there is a unique martingale measure, i.e. the Black-Scholes model is arbitrage free and complete. Therefore every derivative can be uniquely priced, independent of traders risk assumptions, with the formula derived in theorem 2.1.34. The cost for this is that the assumptions are quite rigorous and do not reflect real market situation, one well known observation is the smile and term structure for implied volatilities, contradicting the constant volatility assumptions in the Black-Scholes model.We will investigate this closer in the next chapter.

In this chapter we calculated the delta of a call option and compared it to the calculated price of a call on call option. We proved that a short term call on call option can be approximated by multiplying a call price and a delta. The compound option pricing formula and all other standard results can be found in Etheridge [36]. The definition of the greeks can be found in any suitable textbook, we used Hafner [51], because of the newly introduced second order greeks, DVegaDVol and DDeltaDVol, these are needed in chapter 2.4.

2.2.1 Model Assumptions

In the Black-Scholes model assumption iii, p. 10, is simplified to:

Assumption(BS) iii) The stock is modelled by formula 2.2.1, with positive, real-valued initial stock prices $S_0 \in \mathbb{R}^+$, i.e. $x = S_0 \in \mathbb{R}_+$ and deterministic drift $\mu \in \mathbb{R}$ and positive, deterministic volatility $\nu \in \mathbb{R}^+$.

Formula 2.2.1 Stock Price Process

$$\forall t \in [0, T^*]: \qquad dS_t = \mu S_t dt + \nu S_t dW_t$$

1. Black and Scholes [11].

We therefore only consider one risky asset; the volatility process ν and drift process μ are constant.

2.2.2 Stock Price Distribution

Following the results in chapter 2.1.7 we have that the stock price process is given by:

$$S_t = S_0 \exp\left(\left(\mu - \frac{1}{2}\nu^2\right)t + \nu(W_t - W_0)\right)$$

and that S_t is lognormal distributed with mean $\left(\mu - \frac{1}{2}\nu^2\right)t$ and variance $\nu^2 t$.

2.2.3 Arbitrage Free and Complete Market

Because (EQ 14) in definition 2.1.27 simplifies to $\nu X_t = \mu - r$, the only solution is:

$$X_t = \frac{\mu - r}{\nu}$$

We therefore see that the Black-Scholes model is arbitrage free and complete – Novikov's condition is obviously satisfied. This gives us a unique way to price any derivative, independent of the trader's risk preference and the stock's drift μ. Note that if we would assume a non-risky stock, i.e. a volatility of zero, the market can only be arbitrage free if $\mu = r$.

2.2.4 Martingale Pricing

In the Black-Scholes market model we can also simplify theorem 2.1.34.

Theorem 2.2.2 Black-Scholes Call Pricing Formula

Let $H = (S_T - K)^+$ be a European call option with maturity T and strike K. We have:

$$\Pi_t(H) = S_t \mathbf{N}(d_1) - Ke^{-r(T-t)}\mathbf{N}(d_2)$$

with $\mathbf{N}$ the cumulative normal distribution $\mathbf{N}(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} \exp(-y^2/2)dy$ and

$$d_1 = \frac{\log(S_t/K) + (r + \nu^2/2)(T-t)}{\nu\sqrt{T-t}}, \quad d_2 = d_1 - \nu\sqrt{T-t} \qquad \text{(EQ 1)}$$

where r is the interest rate and ν the stock price volatility.

Proof: See Etheridge [36], pages 118-120. q.e.d.

Using theorem 2.2.2 and the put-call parity (lemma 2.1.36) it is easy to derive a pricing formula for put options. From theorem 2.2.2 we obtain that the price of a European call or put option with fixed interest rate r, expiry T, strike K and volatility ν only depends on the time to maturity $T - t$ and the current stock price, i.e. it is a function of t and S_t, we therefore will write $P : [0, T] \times \mathbb{R}^+ \to \mathbb{R}^+$ with $P(t, S_t) = \Pi_t(H)$. It is easy to see that P is in $\mathcal{C}^{1,2}$ which is enough to apply Itô's lemma to P.

Let (a_t, b_t) be a hedging portfolio for H. Because of theorem 2.1.33 we then have:

$$\forall t \le T \qquad a_t S_t + b_t e^{rt} = \Pi_t(H) = P(t, S_t)$$

Applying Itô's lemma to $P(t, S_t)$ and using formula 2.2.1, we get:

$$\begin{aligned} dP(t, S_t) &= \frac{\partial P}{\partial t} dt + \frac{\partial P}{\partial S} dS_t + \frac{1}{2} \frac{\partial^2 P}{\partial S^2} d\langle S, S\rangle_t \\ &= \left(\frac{\partial P}{\partial t} + \mu S_t \frac{\partial P}{\partial S} + \frac{1}{2} \nu^2 S_t^2 \frac{\partial^2 P}{\partial S^2} \right) dt + \nu S_t \frac{\partial P}{\partial S} dW_t \end{aligned} \qquad \text{(EQ 2)}$$

Using the self-financing property of a hedging portfolio and using formula 2.2.1 we get:

$$\begin{aligned} d(a_t S_t + b_t e^{rt}) &= a_t dS_t + b_t de^{rt} \\ &= (a_t \mu S_t + r b_t e^{rt}) dt + a_t \nu S_t dW_t \end{aligned}$$

Comparing coefficients for W_t yields:

Formula 2.2.3 Hedging Portfolio

$$\begin{aligned} a_t &= \frac{\partial P}{\partial S}(t, S_t) \\ b_t &= e^{-rt}(P(t, S_t) - a_t S_t) \end{aligned}$$

Comparing coefficients for dt yields:

$$(a_t \mu S_t + r b_t e^{rt}) = \frac{\partial P}{\partial t} + \mu S_t \frac{\partial P}{\partial S} + \frac{1}{2} \nu^2 S_t^2 \frac{\partial^2 P}{\partial S^2}$$

Inserting the hedging portfolio from formula 2.2.3, we get

Formula 2.2.4 Black-Scholes PDE

$$rP(t, S_t) = \frac{\partial P}{\partial t} + \frac{1}{2} \nu^2 S_t^2 \frac{\partial^2 P}{\partial S^2} + r S_t \frac{\partial P}{\partial S}$$

with final condition $P(T, S_T) = H(S_T)$.

Since these partial differentials are not just part of formula 2.2.4 but are substantial for risk management we will come across them regularly, therefore each of them will be named.

2.2.5 Black-Scholes Risk Factors (Greeks)

Definition 2.2.5 Greek Functions
The partial differentials of an derivative pricing functions are called *greek functions* or simply just *greeks*. Often one refers to a greek as the greek function evaluated with current values.

Table 2.2.6 Black-Scholes Greeks[ab]

Greek Definition			Call	Put
Delta	δ_t	$\frac{\partial P}{\partial S}$	$\mathbf{N}(d_1) > 0$	$\mathbf{N}(d_1) - 1 < 0$
Gamma	Γ_t	$\frac{\partial^2 P}{\partial S^2}$	$\frac{n(d_1)}{S\nu\sqrt{T-t}} > 0$	
Vega	Λ_t	$\frac{\partial P}{\partial \nu}$	$n(d_1)S(\sqrt{T-t}) > 0$	
Theta	Θ_t	$\frac{\partial P}{\partial t}$	$-\frac{Sn(d_1)\nu}{2\sqrt{T-t}} - rKe^{-r(T-t)}\mathbf{N}(d_2) < 0$	$-\frac{Sn(d_1)\nu}{2\sqrt{T-t}} + rKe^{-r(T-t)}\mathbf{N}(-d_2)$
DVegaD-Vol	Φ_t	$\frac{\partial^2 P}{\partial \nu \partial \nu}$	$S\sqrt{T-t}\,n(d_1)\frac{d_1 d_2}{\nu}$	
DDeltaD-Vol	Ψ_t	$\frac{\partial^2 P}{\partial S \partial \nu}$	$-n(d_1)\frac{d_2}{\nu}$	

a. See Hafner [51], p. 30.
b. For definitions of d_1 or d_2 see (EQ 1).

We plotted Black-Scholes delta by the strike price and the time to expiry in figure 2.2.7. The delta is decreasing for increasing strike price and increasing with increasing time to expiry for a strike price more than 5880. For very low strike prices the delta is first decreasing and then increasing with increasing time to expiry. The delta of a put option is equal to the delta of a call option minus one. As a consequence if we only consider squared delta, the same structure arises for call or put options, i.e. we get increasing delta for increasing in-the-moneyness for call and put options.

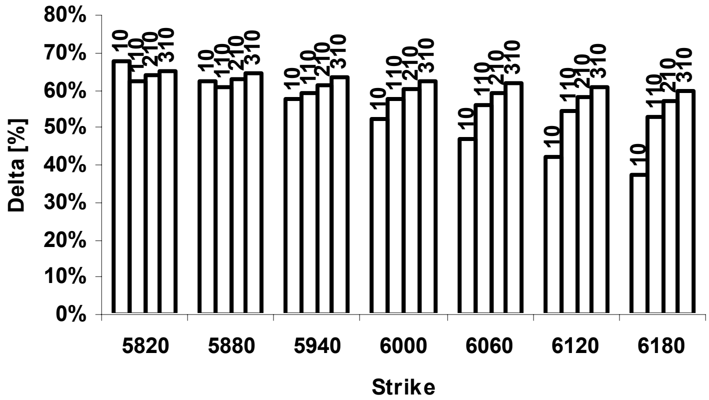


Figure 2.2.7 Black-Scholes Delta

Delta of a call option with spot of 6000 and volatility of 20% for different days to expiry.

2.2.6 Compound Options

Definition 2.2.8 Compound Option

A compound option is an option on another option, that is another option plays the role of the underlying. We therefore have to set two strikes K_0 and K_1 as well as two different expiry dates T_0 and T_1. At time T_0 the holder of the compound option has the right but not the obligation to buy another option (with strike K_1 and expiry T_1) for K_0.

The payoff function is:

$$V(T_0, S_{T_0}) = (P_{T_0}(S_{T_0};T_1, K_1) - K_0)^+ \qquad \text{(EQ 3)}$$

where $P_{T_0}(S_{T_0};T_1, K_1)$ denotes the price process of the underlying option at time T_0.

Theorem 2.2.9 Black-Scholes Call on Call Pricing Formula

Let $H = V(T_0, S_{T_0})$ be a European call on call option with maturity dates T_0 and $T_1 > T_0$ and strikes K_0 and K_1. We have:

$$\Pi_t(H) = e^{-r(T_0 - t)}\int_{x_0}^{\infty}(f(y)\mathbf{N}(\widehat{d_1}(y)) - K_0 e^{-r(T_1 - T_0)}\mathbf{N}(\widehat{d_2}(y)) - K_0)\frac{1}{\sqrt{2\pi}}e^{-y^2/2}dy$$

with $\mathbf{N}$ the cumulative normal distribution, $f(y) := S_0 \exp\left(\nu y\sqrt{T_0 - t} + \left(r - \frac{1}{2}\nu^2\right)(T_0 - t)\right)$, $x_0 := \inf(y \in \mathbb{R} \mid P_{T_0}(f(y);T_1, K_1) \geq K_0)$.

and $$\widehat{d_1}(y) = \frac{\log(S_0/K_1) + \nu y\sqrt{T_0 - t} + rT_1 - \nu^2 T_0 + \frac{1}{2}\nu^2 T_1}{\nu\sqrt{T_1 - T_0}}, \quad \widehat{d_2} = \widehat{d_1} - \nu\sqrt{T - t}$$

Proof: See Etheridge [36], pages 143-144. q.e.d.

In figure 2.2.10 we plotted the price of an at-the-money (ATM) call on call option by the option's underlying strike price and time to expiry. Note that the similar surfaces, i.e. increasing price for increasing time to expiry and increasing price for decreasing strike of the underlying option for short term, ATM call on call options and the delta can be explained by the following fact:

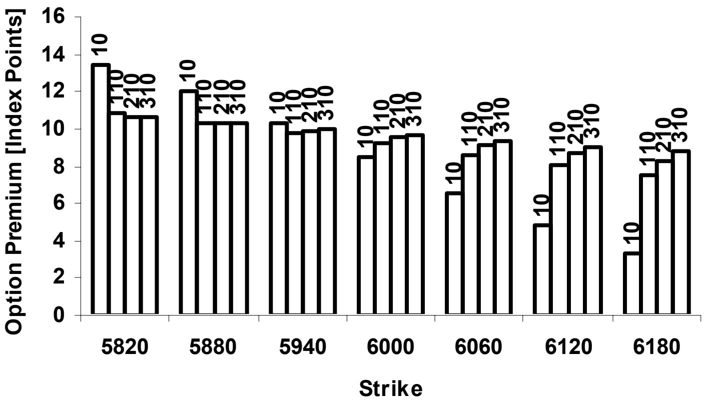


Figure 2.2.10 Black-Scholes Call on Call Option

ATM (i.e. $K_0 = C(S_0, 0; T_1, K_1)$) call on call option with expiry of $T_0 = 6$ hours, volatility of $\nu = 0.2$ and spot of $S_0 = 6000$ by changing time to expiry and strike of the underlying option.

As in (EQ 2) we apply Itô's lemma to $P(t, S_t)$ yielding:

$$\begin{aligned} dP(t, S_t) &= \Theta_t dt + \delta_t dS_t + \frac{1}{2}\Gamma_t d\langle S, S\rangle_t \\ &= \Theta_t dt + \delta_t dS_t + \frac{1}{2}\Gamma_t S_t^2 \nu^2 dt \end{aligned} \qquad \text{(EQ 4)}$$

Because we assumed that T is small, we can assume an interest rate of zero. The Black-Scholes PDE for a call option (formula 2.2.4) therefore simplifies to:

$$\Theta_t + \frac{1}{2}\Gamma_t S_t^2 \nu^2 = 0 \qquad \text{(EQ 5)}$$

Substituting (EQ 5) into (EQ 4) yields:

$$dP(t, S_t) = \delta_t dS_t \qquad \text{(EQ 6)}$$

Again, because of small holding time T we can write:

$$P(T, S_T) - P(0, S_0) = \delta(S_T - S_0) \qquad \text{(EQ 7)}$$

Taking expectations we get:

$$\mathbb{E}_Q((P(T, S_T) - P(0, S_0))^+) = \mathbb{E}_Q(\delta_0(S_T - S_0)^+) = \delta_0 C$$

We therefore can approximate the price of an ATM compound call option by $\delta_0 C$, where δ_0 is the Black-Scholes delta of the underlying option and C is the price of an ATM call option with time to expiry equal to the holding time T. The problem of the constant volatility parameter in the Black-Scholes model assumption leads to a mispricing which is especially strong in the case of an exotic option, like the compound option.

CHAPTER 2.3 Volatility in Financial Markets

In this chapter we will first define two concepts of measuring volatility. The definitions are standard, for example, see Hull [55]. However, the exact mathematical background is often missing, we used Benth [8] as a reference. We calculated historical volatility from DAX closing values and we also used data from traded options at Eurex to derive the implied volatility surface on specific days. After that we will explain newly introduced volatility indices at Deutsche Börse and calculate several indices from data provided by Eurex. We used Carr and Wu [20] as a reference. In chapter 2.3.4 we present one of the obvious and known shortcomings of the Black-Scholes in more detail. This leads us to assume stochastic volatility processes. In chapter 2.3.6 we will give a brief overview of models considered in the past, the main references were Fouque et.al. [46] and Cizek [23]. We close this chapter with a calculation of the volatility of volatility and the volatility indices.

Volatility is a measurement of the degree of fluctuations in financial market. It is defined by:

Definition 2.3.1 Volatility
The *volatility* of a stochastic process X_t is defined to be the square root of its variance:

$$\nu_t := \sqrt{VAR(X_t)}$$

Note that the volatility is the only unknown variable in the Black-Scholes option pricing formula – every other variable is more or less observable.[1] This raises the question of how to get an estimation for the volatility of the underlying. As we have seen in the Black-Scholes framework $\log(S(T)/S(0))$ is normal distributed with variance $\nu^2 T$. Let us assume we observed the stock price process $S(t)$ over one year, i.e. we assume $T = 1$ and the time is measured in years, on $n + 1$ discrete, equidistant points $t_0, \ldots, t_n$ with $t_i - t_{i-1} = \Delta t$, $t_0 = 0$ and $t_n = 1$, setting:

$$R_i := \log\left(\frac{S(t_i)}{S(t_{i-1})}\right)$$

we have that R_i are independent, identical distributed random variables. We hence can apply standard likelihood techniques.[2] Because R_i are also normal distributed we get the following maximum likelihood estimators for the mean:

$$\hat{\eta} = \frac{1}{n}\sum_{i=1}^{n} R_i \qquad \text{(EQ 1)}$$

1. An exact interest rate is not easy to derive, therefore, if available, traded futures are often used.
2. See Benth [8], p. 9 and pp. 15-17.

and for the volatility:[1]

$$\hat{\nu} = \sqrt{\frac{1}{\Delta t \cdot n}\sum_{i=1}^{n}(R_i - \hat{\eta})^2} \quad \text{(EQ 2)}$$

2.3.1 Historical Volatility

Because continuous observation of the underlying movement is impossible the volatility can only be approximated. There are several kinds of volatility approximations.[2] Here we will use daily closing values. Since there are approximately 252 trading days per year we set $\Delta t = 1/252$ in the derived maximum likelihood estimator (EQ 2). The time is collapsed to trading time, i.e. $T = 1$ trading year, because volatility seems to arise mostly from trading and not from the arrival of new information.[3]

Definition 2.3.2 (V[Stock]-HIST) n-day Historical Volatility
Let us assume we observed the closing stock price S_i on n-different days, i.e. $i \in \{1, \ldots, n\}$.
We then define the n-day, annualized *historical volatility* (estimation) by:

$$\hat{\nu} := \sqrt{\frac{252}{(n-1)}\sum_{i=1}^{n}(R_i - \hat{\eta})^2} \text{ with } \hat{\eta} := \frac{1}{n}\sum_{i=1}^{n} R_i$$

and $R_i := \log(S_i/S_{i-1})$ is the log-return of the i-th observed stock price.

We divide by $n-1$ instead of n, resulting in a unbiased estimator. However, as Press et al. points out, we would "probably [be] up to no good anyway" if this difference would matter ([72], p. 611).
Because in the real world volatility is not constant over time, it is common practice to use the n-day historical volatility to price an option with a time to expiry of n-days.

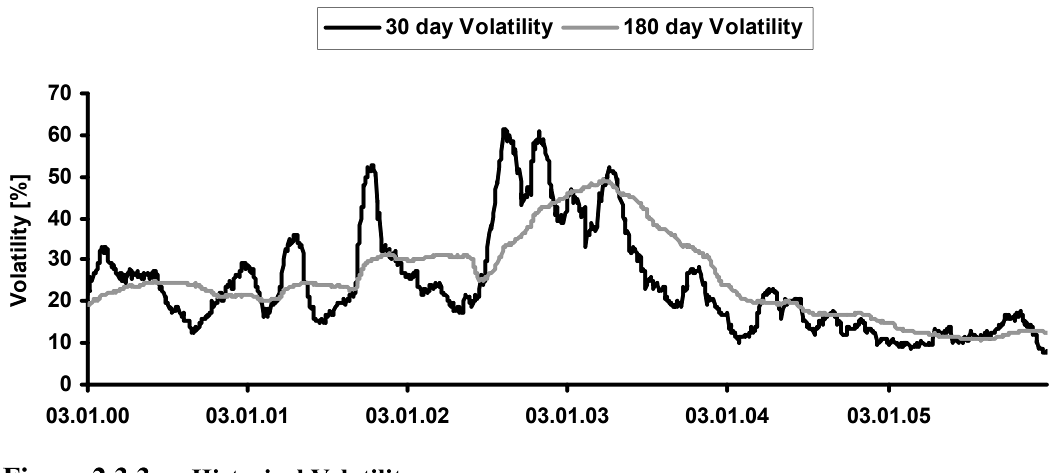


Figure 2.3.3 Historical Volatility
30-day (black) and 180-day (grey) historical volatility, as defined in definition 2.3.2, from DAX closing values.

1. See Benth [8], p. 9 and solution to exercise 1.6, pp. 122-123.
2. See for example Garman [47].
3. This is suggested by investigations on the change in volatility over the weekend; see Hull [55], p. 289.

2.3.2 Implied Volatility

As seen in chapter 2.2 the Black-Scholes price of an option $C(\nu) = C(t, x, r, \nu, K, T)$ is monotone increasing and continuous with the volatility ν. Because any other variable is observable we can define the inverse function σ by $\sigma(C(\nu)) := \nu$, i.e. σ maps the unique option price for each option to the unique volatility that was used to price this option with.[1] This is why traders often quote options in implied volatility rather than the actual option price. The advantage is that it is easier to compare options regarding their implied volatility than the option price, across different markets or at different times.

Note that even though it is appropriate to interpret the implied volatility as a measure for the price, it is neither obvious nor suggested by empirical data, that it is a measure for the futures realized volatility. Rebonato [73] infers, that implied volatility is the wrong number to put in the wrong formula to obtain the right price.[2]

Definition 2.3.4 (IV) Implied Volatility Surface

We call $\sigma_t(K, T) = \sigma(t, x, r, C^{obs}, K, T)$ the *implied volatility* for the K-call (put) T option with premium C^{obs} at time t. Furthermore we call σ_t a volatility surface, where σ_t is defined by:

$$C(t, S_t, r, \sigma_t(t, S_t, r, C^{obs}, K, T), K, T) = C^{obs} \qquad \text{(EQ 3)}$$

where C^{obs} is the observed option price for an option with strike K, expiry T at time t.

Note that because of put-call parity (lemma 2.1.36) the implied volatility of a European call option must be the same as from a European put option with the same strike and maturity.[3] We therefore do not distinguish between $\sigma_t(K, T)$ derived from a call option or from a put option.

Unfortunately, no closed solution is available for the function $\sigma_t(K, T)$. As a consequence approximation formulas like Newton's algorithm for finding roots has to be applied to $\nu \to C^{obs} - C(\nu)$, where C^{obs} is the observed option price and $C(\nu)$ is the Black-Scholes option price with volatility ν and the other variables as used for the option C^{obs}.

Even though in the Black-Scholes model a constant volatility is assumed throughout all options, i.e. $\sigma_t(K, T) = c \in \mathbb{R}^+$ for all available strikes K and maturities T and all $t \le T$ empirical observations (from traded options) show different results.[4]

1. Observed implied volatility surfaces often show a smile pattern (figure 2.3.5[5]) or a skew pattern (figure 2.3.6), i.e. $\sigma_t(K_1, T) \neq \sigma_t(K_2, T)$ for $K_1 \neq K_2$.[6]
2. Also changes in implied volatility for different maturities can be observed, i.e. $\sigma_t(K, T_1) \neq \sigma_t(K, T_2)$ for $T_1 \neq T_2$.This phenomenon is called the term structure of the implied

1. Such an implied volatility can be found if the observed option price satisfies $C^{obs} > C(t, x, r, 0, K, T)$, i.e. the observed option price must be higher than the Black-Scholes price, priced with 0 volatility.
2. See Rebonato [73], p. 78.
3. Certain problems may arise, like different tax rates for dividends; see Hafner [51], p. 78.
4. See Dumas et. al. [32] or Bolek [12] for a comprehensive study of DAX implied volatility, pp.122-133.
5. Since the market crash in 1987 the minimum of the curve is often observed to be OTM for calls and ITM for puts, i.e. a premium is charged for these kinds of options because of market crash phobia (see Hull [55], p. 381).
6. See Hafner [51], p. 39.

volatility shape. The term structure can have very different shapes, it can be increasing (figure 2.3.7), decreasing (figure 2.3.8) or constant by increasing time to expiry. This feature can also be observed from historical volatilities.[1] Also note that the shape of the term structure seems to be influenced by the level of the volatility, which can be explained by the fact that during market turmoils the near future is much more uncertain than the distant one.[2]

3. The last observation shows that the implied volatility from one option is different at different times, i.e. $\sigma_t(K, T) \neq \sigma_s(K, T)$ for $t \neq s$.

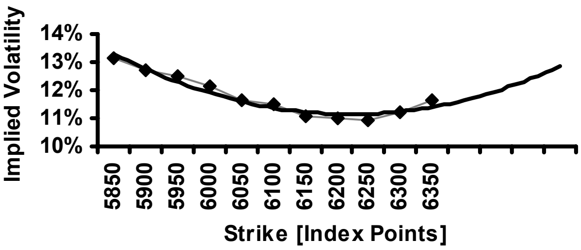


Figure 2.3.5 Implied Volatility Smile

Implied volatility smile (approximation: fat line), from traded DAX call options (squares) on 7 April 2006, expiring in April. DAX close: 5,952.92.

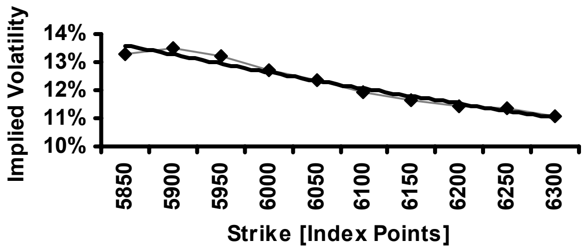


Figure 2.3.6 Implied Volatility Skew

Implied volatility skew (approximation: fat line), from traded DAX call options (squares) on 5 April 2006, expiring in April. DAX close: 6,029.20.

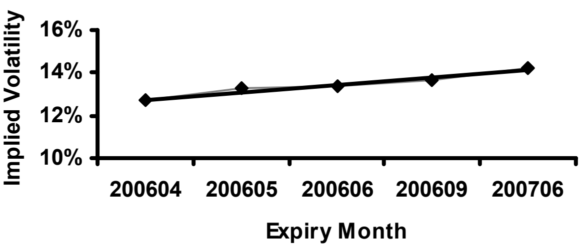


Figure 2.3.7 Implied Volatility Term Structure

Implied volatility term structure (approximation: fat line), from traded DAX call options (squares) on 7 April 2006, with strike 6000. DAX close: 5,952.92.

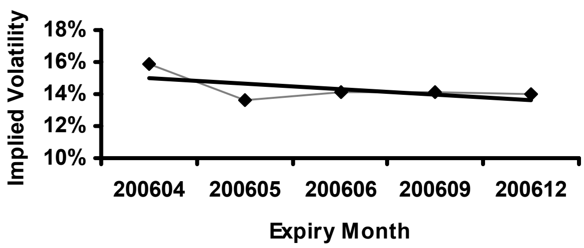


Figure 2.3.8 Implied Volatility Term Structure

Implied volatility term structure (approximation: fat line), from traded DAX call options (squares) on 19 April 2006, with strike 6000. DAX close: 5,993.76.

These results can best be expressed not in terms of strike and maturity, but in moneyness M and time to expiry τ, for example observations 2 and 3 are both showing a dependency of τ. Moneyness should be a measure of how valuable the contract is, i.e. of how far the contract is in-the-money (ITM). The moneyness should therefore be a function of the strike price, the underlying stock price and also the time to maturity and the interest rate, because for example the probability of a 10% stock increase is much higher (depending on the interest rate r) over two years than over one week. We could therefore reasonably call an option with time to expiry of two years and a strike of $(1 + 1/10)S_0$ with a spot of S_0 at-the-money (ATM), but not so the same option with time to expiry of just one week. We would rather call it out-of-the-money (OTM).[3]

---

1. See figure 2.3.3.
2. See also figure 2.3.14.
3. See Hafner [51] p. 85.

Definition 2.3.9 Valid Moneyness[1]

Every function $\mathbf{m} \in C^2(\{(x_i)_{i<5} \in [0, T^*] \times \mathbb{R}^+ \times \mathbb{R}^+ \times (0, T^*] \times \mathbb{R}_0^+ \mid x_3 \geq x_0\})$ satisfying:

- $\lim_{t \to T} \mathbf{m}(t, s, K, T, r) < \infty$, $\mathbf{P}$-almost surely,
- $\lim_{t \to T} \frac{\partial}{\partial t}\mathbf{m}(t, s, K, T, r) < \infty$, $\mathbf{P}$-almost surely and
- $\mathbf{m}(t, s, K, T, r)$ is increasing with respect to K

is called a *valid moneyness* function. We set $M = M_t = \mathbf{m}(t, S_t, K, T, r)$ and call M a valid moneyness.

With respect to a valid moneyness, we can define a relative volatility surface:

Definition 2.3.10 Relative Implied Volatility Surface

We call $\tilde{\sigma}_t(M, \tau) = \tilde{\sigma}(t, S_t, r, C^{obs}, \mathbf{m}(t, S_t, K, T, r), \tau)$ a *relative volatility surface*, where $M = \mathbf{m}(t, S_t, K, T, r)$ is a valid moneyness and $\tau = T - t$ is the time to expiry of an option with expiry T, strike K and premium C^{obs}. And $\tilde{\sigma}$ defined by:

$$C(t, S_t, r, \tilde{\sigma}(t, S_t, r, C^{obs}, \mathbf{m}(t, S_t, K, T, r), \tau), K, T) = C^{obs}.$$

Because of the last assumption in definition 2.3.9 we can invert $\mathbf{m}(t, S_t, K, T, r)$ with respect to K and therefore have the following relation between the absolute (definition 2.3.4) and the relative implied volatility surface (definition 2.3.10):

$$\tilde{\sigma}_t(M, \tau) = \sigma_t(\mathbf{m}^{-1}(M), t + \tau) \qquad \text{(EQ 4)}$$

One possible example of a valid moneyness function is the log-simple moneyness function:

Definition 2.3.11 Log-Simple Moneyness Function

If we set with $\mathbf{m}(t, s, K, T, r) := \log\left(\frac{K}{se^{r(T-t)}}\right)$ in definition 2.3.9 we call $\mathbf{m}$ the *log-simple moneyness* function. It can be easily proved that $\mathbf{m}$ is in fact a valid moneyness function.

Using the log-simple moneyness we hence name a call or put option with strike K and expiry T, ATM if we have $\mathbf{m}(t, S_t, K, T, r) \approx 0$. If we have $\mathbf{m}(t, S_t, K, T, r) < 0$ we label a call and put option ITM and OTM, respectively. In case of a positive moneyness $\mathbf{m}(t, S_t, K, T, r) > 0$, we say that the call option is OTM and the put option is ITM.

Arbitrage Free Markets

Even though we saw that the implied volatility is far from being constant, no-arbitrage arguments imply that we can find an upper and lower bound for the slope of the implied volatility smile.[2]

1. See Hafner [51] p. 61
2. See Fouque [46], pp. 35-37.

Differentiating (EQ 3) with respect to the strike K yields:

$$\frac{\partial}{\partial K}C^{obs} = \frac{\partial C}{\partial K} + \frac{\partial C}{\partial \nu} \cdot \frac{\partial \sigma}{\partial K} \qquad \text{(EQ 5)}$$

Because of no-arbitrage arguments call prices must be decreasing in the strike K, i.e. $\frac{\partial}{\partial K}C^{obs} \le 0$ and because the vega $\frac{\partial C}{\partial \nu}$ is always positive, we can deduce from (EQ 5) for a call option:

$$\frac{\partial \sigma}{\partial K} \le -\frac{\partial C / \partial K}{\partial C / \partial \nu}$$

And in analogy for a put option:

$$\frac{\partial \sigma}{\partial K} \ge -\frac{\partial P / \partial K}{\partial P / \partial \nu}$$

which in the Black-Scholes framework becomes:

$$-\frac{\sqrt{2\pi}}{x\sqrt{T-t}} \cdot (1 - \mathbf{N}(d_2))e^{-r(T-t) + d_1^2/2} \le \frac{\partial \sigma}{\partial K} \le \frac{\sqrt{2\pi}}{x\sqrt{T-t}} \cdot \mathbf{N}(d_2)e^{-r(T-t) + d_1^2/2}$$

Stock Price Distribution

Because we have a one-to-one relation between the option price and the implied volatility, we could also express the formula for the risk-neutral density (see EQ 21, p. 27) only in terms of the implied volatility smile $\sigma_t(K, T)$.[1]

2.3.3 VDAX-NEW: a Measurement for Volatility

Construction of Subindices

Using the definition of a variance swap definition 2.1.9 and the derived pricing formula in lemma 2.1.39 we can define a subindex V_t^i for each expiry series[2] T_i of DAX options representing the variance delivery price $K_t(T) = \mathbb{E}_Q[w_t(T)]$, where $w_t(T) = \frac{1}{T-t}\int_t^T \nu_t^2 dt$. Because there is no continuum of traded options and we cannot set S_* to S_0e^{rt} in (EQ 30, p. 32) we get:[3]

1. See Hafner [51], p. 52.
2. See descriptions of DAX options at Eurex on page 77.
3. See Carr and Wu [20], p. 28.

$$K_t(T) = \frac{2}{\tau}\left[r\tau - \log\left(\frac{S_*}{S_0}\right) - \frac{(S_0e^{r\tau} - S_*)}{S_*} + e^{r\tau}\int_{S_*}^{\infty}\frac{C_t(K,T)}{K^2}dK + e^{r\tau}\int_0^{S_*}\frac{P_t(K,T)}{K^2}dK\right]$$

$$\approx \frac{2}{\tau}\sum_{i=0}^{n}\frac{P_t(K_i,T)}{K_i^2}\Delta K_i + \frac{2}{\tau}\sum_{i=n+1}^{m}\frac{C_t(K_i,T)}{K_i^2}\Delta K_i - \frac{1}{\tau}\left(\frac{S_0e^{r\tau} - K_n}{K_n}\right)^2 \qquad \text{(EQ 6)}$$

where $\tau := T - t$ the time to expiry, $K_n = \max\{K_i < S_0e^{rt}\}$ is defined to be the strike right below the future price, $C_t(K, T)$ and $P_t(K, T)$ the price at time t for a call and put, respectively, with strike K and maturity T, and where we made use of a Taylor approximation for $\log(x)$

$$r\tau - \log\left(\frac{S_*}{S_0}\right) - \frac{(S_0e^{r\tau} - S_*)}{S_*}$$

$$= \log(e^{r\tau}) - \log\left(\frac{S_*}{S_0}\right) - \frac{(S_0e^{r\tau} - S_*)}{S_*}$$

$$= \log\left(\frac{S_0e^{r\tau}}{S_*}\right) - \frac{(S_0e^{r\tau} - S_*)}{S_*}$$

$$= -\frac{(S_0e^{r\tau} - S_*)}{S_*} + \left(\frac{S_0e^{r\tau}}{S_*} - 1\right) - \frac{1}{2}\left(\frac{S_0e^{r\tau}}{S_*} - 1\right)^2 + \ldots \approx -\frac{1}{2}\left(\frac{S_0e^{r\tau}}{S_*} - 1\right)^2$$

The value of each Subindex is now given by:

$$V_t^i = 100\sqrt{K_t(T_i)}$$

Construction of Main Indices

Definition 2.3.12 (V[Stock]-NEW) n-day Volatility Index[1]
The *volatility index* is interpolated from the two subindices including the expiry of n days, or extrapolated from the two closest if time to expiry from the first subindex is higher than n days. Official VDAX-NEW as published by Deutsche Börse is defined to be 30-day VDAX-NEW.[2]

Note that lemma 2.1.39 did not need any special requirements for the volatility process and therefore it does not matter if we price VDAX-NEW using Black-Scholes model or any other financial market model (where assumption ii is fulfilled). However, some problems occur from the fact that in reality options are not available for every strike.

1. For details on VIX, which is the newly introduced volatility index on the S&P 500, and a comparison to VXO, the old volatility index on the S&P 100, see Carr and Wu [20]. The definitions of the VIX and the VXO are the same as for VDAX-NEW and VDAX, respectively.
2. For a verification of the correctness of our own calculations see figure A.6.

So let us take a closer look at the calculation of 90-day VDAX-NEW. As can be seen in figure 2.3.13 the 90-day VDAX-NEW is in general calculated from the third and fourth subindex. Let us see which expiry months are available in May before and after the expiry day in May.[1] Before the expiry day they are, May, June, July and September, and after the expiry day they are June, July, August and September. We therefore see that the third subindex is calculated from options with a newly introduced expiry month. Obviously the amount of different strikes from quoted options for a newly introduced expiry day is much lower than the number of strikes available for other expiry days. As a consequence we take the sum in (EQ 6) over less (positive) values and hence the subindex will be lower. Which explains the drops in figure 2.3.14 for 90-day VDAX-NEW. Similar model specific limitations may also be responsible for the weekday effect of newly introduced volatility indices, i.e. having an average value on Fridays less than on other weekdays.[2]

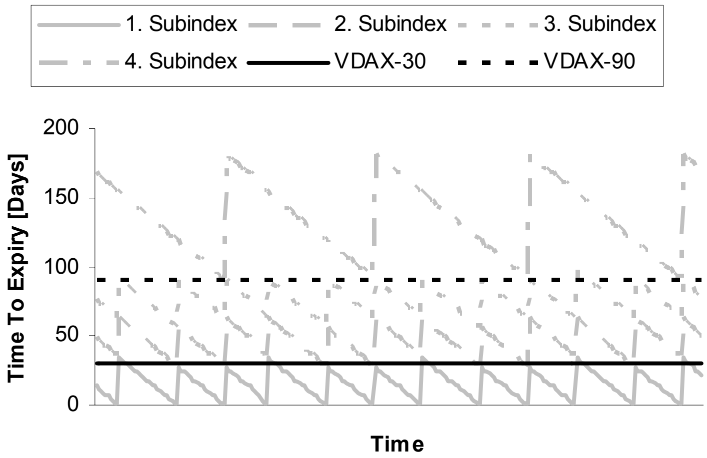


Figure 2.3.13 Relevant Sub-Indices

Relevant Sub-Indices 1, 2, 3 and 4 for 30 (solid, black) and 90 (dotted, black) day VDAX-NEW

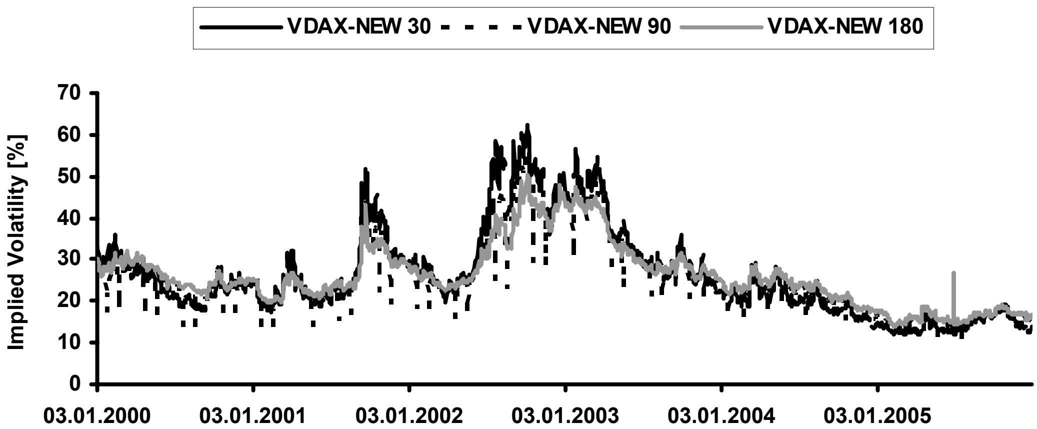


Figure 2.3.14 VDAX-NEW

Long term (grey) and short term (black) VDAX-NEW comparison.

1. Footnote 1, p. 77 describes the available months.
2. The average value on Fridays in 1999 to 2005 was about 0.5 lower than on Mondays, for VDAX-NEW, VSMI and VESX.

2.3.4 Why NOT Use a Geometric Brownian Motion For Stock Modelling?

As we have seen on page 25 a deterministic drift and volatility process automatically leads to a geometric Brownian motion model. The only arguments we found in chapter 2.1.8 were that the stock price has to be rough, or else the market cannot be arbitrage free.

The implied volatility surface suggests that a geometric Brownian motion is not adequate to model stock price processes. The market seems to expect big differences in the stock price more likely to appear than predicted by the lognormal distribution (volatility smile). This can also be verified by empirical data, the distribution of stock returns has usually fatter tails than a normal distribution, i.e. it has a positive kurtosis (fourth moment).[1] The volatility surface (volatility skew) also implies that the market expects decreases in the underlying more often than predicted by the Black-Scholes model. This results in an asymmetric distribution of underlying prices, which we can also observe in empirical data, resulting in a negative skewness (third moment), see figure 2.3.15.[2]
We therefore can conclude that within our model assumptions a deterministic volatility process is not sufficient to describe the observed distribution. Instead of changing the volatility process we could also relax our model assumptions; for example, we could allow the stock to make jumps, resulting in a discontinuous model[3] or, instead of using a Brownian motion as the driving process of the underlyings, we could use a fractional Brownian motion[4]. An overview of possible explanations for the observed smile surfaces can be found in Hafner [51], page 44.

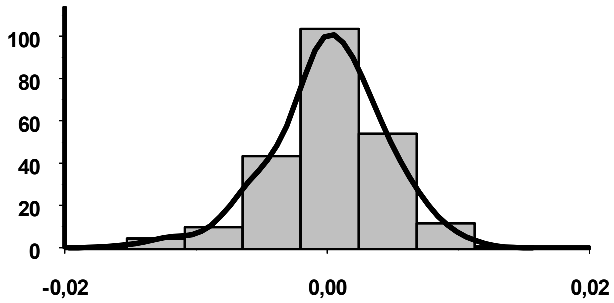


Figure 2.3.15 DAX Distribution

Distribution of log-returns from DAX closing values in 2005. (resulting in a kurtosis of 1 and a skewness of -0.3)

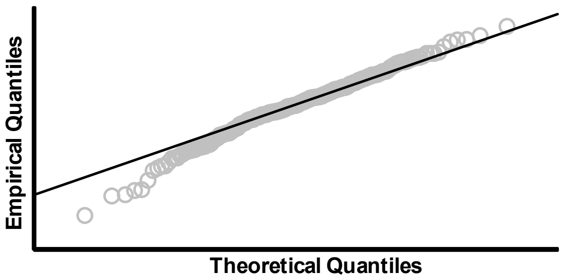


Figure 2.3.16 DAX Quantiles

A QQ-Plot shows that low log-returns occur more often and high ones less than predicted by a normal distribution (solid line).

We will now try to establish a model with smile surfaces by changing the volatility process.

2.3.5 Time-Dependent, Deterministic Volatility

In chapter 2.3.2, observations 2 and 3 showed that the implied volatility is changing with time to expiry. The Black-Scholes assumption of constant volatility can therefore be relaxed by allowing, that

1. For a definition of skewness and kurtosis see Press et. al. [72] p. 612.
2. An empirical observation on FTSE and NASDAQ can be found in Benth [8], pp. 11-31.
3. For a short introduction see for example Etheridge [36], pp. 175-180.
4. An introduction of stochastic calculation with fractional Brownian motion is given in Biagini and Øksendal [9].

the volatility is time dependent. If we would assume a deterministic, time dependent volatility process in the general setup, i.e. $\nu : \mathbb{R}^+ \to \mathbb{R}^+$, we know from chapter 2.1.7 that the stock prices are lognormal distributed, with mean $\left(\mu - \frac{1}{2}\overline{\nu^2(t)}\right)t$ and variance $\overline{\nu^2(t)} \cdot t$, where $\overline{\nu^2(t)} = \frac{1}{t}\int_0^t \nu^2(s)ds$.[1] Therefore the Black-Scholes option pricing formula theorem 2.2.2 is still valid with ν substituted by the expected, root-mean-square volatility $\mathbb{E}_Q[\sqrt{\overline{\nu^2(t)}}]$ over the options lifetime. This is measured by the subindices defined in chapter 2.3.3.

We can therefore simulate the term-structure of the implied volatility surface.

2.3.6 Stochastic Volatility

Let us write $\nu_t = f(Y_t)$ for the volatility process driving the underlying, and let Z_t be a Brownian motion correlated with the Brownian motion driving the stock by $d\langle W, Z\rangle_t = \rho dt$. Note that if $|\rho| \neq 1$, we have two independent Brownian motions W_t and $\hat{Z}_t := \rho W_t + \sqrt{1-\rho^2}Z_t$. Because volatility is not tradeable we then have an incomplete market model, and the no-arbitrage price is in general not unique. For example let the volatility be bounded by $\nu_t \in [\nu_{min}, \nu_{max}]$ the European call option price in a stochastic volatility model V_t is bounded by the Black-Scholes prices, i.e. $V_t \in [C_{BS}(\nu_{min}), C_{BS}(\nu_{max})]$, if volatility is not bounded the borders are the stock price and the intrinsic value of the option $V_t \in [(S_t - K)^+, S_t]$ and the borders will be attained for some equivalent martingale measure.[2] To get a unique no-arbitrage option price (theorem 2.1.34) one has to specify a market price of risk process (definition 2.1.27), i.e. the risk preference of the investors.

Several different processes have been consider, among these are:

- lognormal

$$dY_t = \alpha Y_t dt + \beta Y_t dZ_t$$

- Ornstein-Uhlenbeck (OU)

$$dY_t = \alpha(m - Y_t)dt + \beta dZ_t$$

- Feller or Cox-Ingersoll-Ross (CIR)

$$dY_t = \alpha(m - Y_t)dt + \beta\sqrt{Y_t}dZ_t$$

The influence of the volatility of variance parameter β and the correlation of the volatility process with the stock price process ρ in the Heston model, is that β defines the convexity of the implied volatility smile and ρ defines its skewness.[3]

1. See Fouque et. al. [46], pp. 38-39.
2. See Fouque et.al. [46], p. 49. See also footnote 2, p. 71.
3. A good analysis can be found in Cizek et. al. [23], p. 177, see also chapter 2.4.2.

Table 2.3.17 Models of Volatility[a]

Authors	Correlation	f(y)	Y Process
Hull-White	$\rho = 0$	$f(y) = \sqrt{y}$	Lognormal
Scott	$\rho = 0$	$f(y) = \exp(y)$	Mean-Reverting OU
Stein-Stein	$\rho = 0$	$f(y) = \|y\|$	Mean-Reverting OU
Ball-Roma	$\rho = 0$	$f(y) = \sqrt{y}$	CIR
Heston	$\rho \neq 0$	$f(y) = \sqrt{y}$	CIR

a. See Fouque et. al. [46], p. 42.

Perfect Correlation $|\rho| = 1$

Let us assume $Y_t = (t, S_t)$. Several different volatility surfaces $f(Y_t)$ where considered in the past, among these are the constant elasticity of variance diffusion model by Cox [25]:

$$\nu_t = f(t, x) = vx^c$$

with constants $v > 0$ and $0 \le c < 1$. Where the name arises from the fact, that $\frac{\partial f}{\partial x}(t, S_t) \cdot \frac{S_t}{f(t, S_t)} = c$ is constant.

Another approach, called the implied tree approach, was proposed by Dupire [33]. It can be shown that if a continuum of call option prices $C_t(K, T)$ can be observed, we can derive ν with:

$$\nu(K, T) = \sqrt{2\frac{\frac{\partial C_0}{\partial T}(K, T) + rK\frac{\partial C_0}{\partial K}(K, T)}{\frac{\partial^2 C_0}{\partial K^2}(K, T)}}$$

But unfortunately a correlation of 1 or -1 cannot be verified by empirical observations, which can explain the poor forecasting results found by Dumas et. al. [32], but observations show that a negative correlation can be found for options on stocks or stock indices. This observation is called the leverage effect.[1]

1. See Bolek [12], pp. 58-61.

No Correlation $\rho = 0$

If we would assume a correlation of 0 between the volatility process and the underlying[1], i.e. $<Y_t, S_t> = 0$, we can further simplify theorem 2.1.34, by:

$$\begin{aligned}\Pi_t(H) &= S_{0,t}\mathbb{E}_Q\left[\frac{H}{S_{0,T}} \mid \mathcal{F}_t\right] \\ &= \mathbb{E}_Q\left[S_{0,t}\mathbb{E}_Q\left[\frac{H}{S_{0,T}} \mid \mathcal{F}_t, \nu_s, t \le s \le T\right] \mid \mathcal{F}_t\right] \\ &= \mathbb{E}_Q\left[C_{BS}(t, x; K, T; \sqrt{\overline{\nu^2(t)}}) \mid Y_t = y\right]\end{aligned}$$

(where the last equation is following, if we set the volatility process Y_t to be Markov) the theoretical price is therefore given by the Black-Scholes price averaged over all possible volatility paths,

$$\overline{\nu^2(t)} = \frac{1}{T-t}\int_t^T f(Y_s)^2 ds \qquad \text{(EQ 7)}$$

This formula is called the Hull-White Formula, see Fouque et. al. [46] page 51. In this framework we can prove that our intention, to model volatility smiles, is in fact realized:

Theorem 2.3.18 Renault Touzi

Let the volatility process ν_t be independent to the Brownian motion driving the underlying W_t. Suppose further that the market price of volatility risk process is a function of t and Y_t, but not of S_t, and that the root-mean-square time average volatility $\overline{\nu^2(t)}$, defined in (EQ 7), is a square integrable random variable, than the implied volatility surface $\sigma_t(K, T)$ smiles in K, i.e. for fixed t and T the curve $K \to \sigma_t(K, T)$ is locally convex around the minimum $K_{min} = S_t \exp(r(T-t))$, the forward stock price.

Proof: See Fouque et. al. [46], pages 52-53. q.e.d.

2.3.7 Volatility of Volatility

Instead of calculating the volatility of the underlying, we can also calculate the volatility of historical volatility (definition 2.3.2) or implied volatility (definition 2.3.12) by substituting the closing stock price S_i in definition 2.3.2 by the n-day VDAX-HIST or the closing price of n-day VDAX-NEW.
Note however that the volatility process is not lognormal distributed, therefore the maximum likelihood estimators (EQ 1) and (EQ 2) are not correct anymore. But as Benth [8], page 70, points out, these formulas are generally used for all kind of processes.

1. Which seems appropriate for foreign exchanges, see Fouque et. al. [46], p. 51.

It is interesting to note that the term structure of both volatilities are almost always decreasing, whereas the term structure of implied or historical volatility of the underlying can have very different shapes.[1]

The exceptions to this observation from the volatility of VDAX-NEW, arise from the technical difficulties mentioned on page 46, we therefore have not plotted the volatility of 90-day VDAX-NEW in figure 2.3.19.

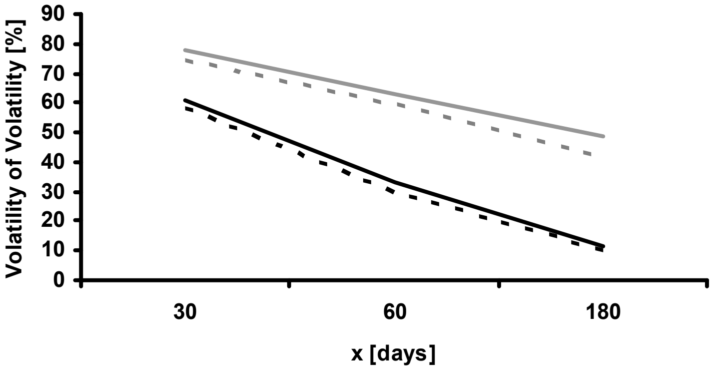


Figure 2.3.19 Volatility of Volatility Term Structure

30-day and 180-day volatility from x-day VDAX-NEW (dotted, grey) and (solid, grey), respectively. 30-day and 180-day volatility from x-day VDAX-HIST (dotted, black) and (solid, black), respectively, from 1.1.2000 to 31.12.2005.

1. The daily term structure of volatility of VDAX-NEW was decreasing almost 90% of all days, from volatility of VDAX-HIST in almost 100%. In contrast to the daily term structure of historical volatility, which was decreasing in about 25%. For details of the exact definition of the term structure and the time span, see figure 2.3.19.

CHAPTER 2.4 Stochastic Implied Volatility Model

Instead of modelling the (instantaneous) volatility of the stock ν_t another approach is to model the dynamics of the implied volatility surface $(t, K, T) \rightarrow \sigma_t(K, T)$ or parts of it, the volatility smile $(t, K) \rightarrow \sigma_t(K, T)$, volatility term structure $(t, T) \rightarrow \sigma_t(K, T)$ or just one single implied volatility $(t) \rightarrow \sigma_t(K, T)$. This was started by Lyons [66] in 1997, who modelled the stochastic evolution of a single implied volatility. Others followed modelling the implied volatility.[1] One important result was derived by Albanese et al. [2], who showed that in a stochastic implied volatility model, the instantaneous volatility of the stock ν_t can be expressed as the limit of implied volatility from at-the-money options. The main advantages of this approach are that because we have $n \leq d$, i.e. we have more traded assets d than random sources n the market model is in general complete. Also by construction these models are free from static arbitrage opportunities, as defined by Carr et al. [18]. The problem of an estimation of the unobservable volatility is avoided by using observable option prices. The main references for this chapter are Hafner [51] and Schönbucher [79].
We start by defining the model assumptions and following Hafner [51] to derive a process of the modelled call option prices under the real world measure and under the risk neutral measure. Using so called no-bubble restrictions from Schönbucher [79] we get a polynomial stating the relationship between the implied volatility and the stock price volatility. Using standard techniques we will approximate the risk neutral distribution with a butterfly spread, see Carr and Madan [19], and calculate it from a volatility smile implied by the polynomial. This relationship will be made clearer in the following, where we prove (following Hafner [51]) that the ATM implied volatility converges against the stock price volatility. This directly results into a no-arbitrage pricing formula. At the end of this chapter we will summarize the results from Hafner [51] of a model calibration.

2.4.1 Model Assumptions

Primary traded securities are a money marker account B_t, one non-dividend paying stock S_t and a continuum of European call options on the stock, i.e. we have a call option price $C_t(K, T)$ for every strike $K \in \mathbb{R}^+$ and every time to expiry $T \leq T^*$ at every possible time $t \in [0, T]$.
We assume the same conditions as in the general setup (see chapter 2.1.2) with one stock ($d = 1$) and a $p+1$-dimensional Brownian Motion $(W_0, \ldots, W_p) = W : [0, T^*] \times \Omega \rightarrow \mathbb{R}^{p+1}$.

The stock will just depend on the first component of the Brownian motion:

$$\forall t \in [0, T^*]: \qquad dS_t = \mu_t S_t dt + \nu_t S_t dW_{0,t}$$

1. See Hafner [51], p. 55.

We further assume that we can represent every option with p many risk factors:

Assumption iv) Let there be a function $g(t, M, \tau, y_1, \ldots, y_p) \in \mathcal{C}^2([0, T^*] \times \mathbb{R} \times (0, T^* - t] \times \mathbb{R}^p)$ that completely describes the relative volatility surface $\tilde{\sigma}(M, \tau)$ (definition 2.3.10)

$$\tilde{\sigma}(M, \tau) = g(t, M, \tau, Y_{1,t}, \ldots, Y_{p,t}) \qquad \text{(EQ 1)}$$

where each risk-factor Y_i is assumed to be an Itô process (definition 2.1.2):

$$dY_{i,t} = \alpha_{i,t}dt + \sum_{j=0}^{p} \gamma_{i,j,t}dW_{j,t} \qquad \text{(EQ 2)}$$

Real World Model

Following our strategy derived in chapter 2.1, we will first try to find an equivalent martingale measure. From theorem 2.1.30 we know that every equivalent measure is coming from a Girsanov transformation. Let us assume we have a $\mathbf{F}$-predictable stochastic process $X = (X_0, \ldots, X_d)$, with $\mathcal{E}(X \bullet W)_t$ being a martingale, hence the Girsanov theorem gives us an equivalent measure $\mathbf{Q}$ and a new $d+1$-dimensional $\mathbf{Q}$-Brownian motion:

$$W_t^* = W_t + \int_0^t X_s ds$$

It is easy to show that

$$X_{0,t} = \frac{\mu_t - r}{\nu_t} \qquad \text{(EQ 3)}$$

must be satisfied, to make the discounted underlying stock S^* a $\mathbf{Q}$-local martingale. The question remains if we can find stochastic processes X_i for $i \in \{1, \ldots, d\}$, so that all discounted call option price processes $C_t^*(K, T)$ are local martingales as well. This is necessary if $\mathbf{Q}$ should be an equivalent martingale measure.

Let us begin with finding the dynamics for the call price process with fixed strike K and time to expiry T. Applying Itô's lemma to the Black-Scholes call option pricing formula $\Pi_t((S_T - K)^+)$ (theorem 2.2.2), where we understand $C_t(K, T) := \Pi_t(H)$ as a function of the time t, the underlying price S_t and volatility $\sigma_t(K, T)$, we get

$$\begin{aligned} dC_t(K, T) = {} & \Theta_t dt + \delta_t dS_t + \frac{1}{2}\Gamma_t d\langle S, S\rangle_t + \Lambda_t d\sigma_t \\ & + \frac{1}{2}\Phi_t d\langle \sigma, \sigma\rangle_t \\ & + \frac{1}{2}\Psi_t d\langle S, \sigma\rangle_t \end{aligned} \qquad \text{(EQ 4)}$$

where we wrote σ_t for $\sigma_t(K, T)$ and set the Black-Scholes greeks[1]

$$\begin{aligned}
\delta_t &= \delta_{BS}(t, S_t, K, T, r, \sigma_t(K, T)) \\
\Gamma_t &= \Gamma_{BS}(t, S_t, K, T, r, \sigma_t(K, T)) \\
\Lambda_t &= \Lambda_{BS}(t, S_t, K, T, r, \sigma_t(K, T)) \\
\Theta_t &= \Theta_{BS}(t, S_t, K, T, r, \sigma_t(K, T)) \\
\Phi_t &= \Phi_{BS}(t, S_t, K, T, r, \sigma_t(K, T)) \\
\Psi_t &= \Psi_{BS}(t, S_t, K, T, r, \sigma_t(K, T))
\end{aligned}$$

To further simplify (EQ 4) we need the dynamics of the (absolute) implied volatility surface $\sigma_t(K, T)$ (see definition 2.3.4). To derive this we proceed in four steps:[2]

1. First we write a SDE for the function that describes the relative implied volatility surface g from (EQ 1). Because we assumed it would be twice differentiable we can apply Itô's lemma to $g(t, Y_{1,t}, \ldots, Y_{p,t}) = g(t, M, \tau, Y_{1,t}, \ldots, Y_{p,t})$ for any fixed moneyness M and time to expiry τ:

$$\begin{aligned}
dg(t, Y_{1,t}, \ldots, Y_{p,t}) &= \frac{\partial g}{\partial t}(t, Y_{1,t}, \ldots, Y_{p,t})dt \\
&+ \sum_{i=1}^{p} \frac{\partial g}{\partial y_i}(t, Y_{1,t}, \ldots, Y_{p,t})dY_{i,t} \\
&+ \sum_{i=1}^{p} \sum_{k=1}^{p} \frac{\partial^2 g}{\partial y_i \partial y_k}(t, Y_{1,t}, \ldots, Y_{p,t})d\langle Y_i, Y_k\rangle_t
\end{aligned} \qquad \text{(EQ 5)}$$

using the dynamics of the risk factors (EQ 2) yields $d\langle Y_i, Y_k\rangle = \sum_{j=0}^{p} \gamma_{i,j,t}\gamma_{k,j,t}dt$, we can hence write (EQ 5) as:

$$dg(t, Y_{1,t}, \ldots, Y_{p,t}) = \tilde{\eta}_t dt + \tilde{\vartheta}_t dW_t \qquad \text{(EQ 6)}$$

with processes:

$$\begin{aligned}
\tilde{\eta}_t &= \frac{\partial g}{\partial t}(t, Y_{1,t}, \ldots, Y_{p,t}) + \sum_{i=1}^{p} \alpha_{i,t} \frac{\partial g}{\partial y_i}(t, Y_{1,t}, \ldots, Y_{p,t}) \\
&+ \frac{1}{2} \sum_{i=1}^{p} \sum_{k=1}^{p} \sum_{j=0}^{p} \gamma_{i,j,t}\gamma_{k,j,t} \frac{\partial^2 g}{\partial y_i \partial y_k}(t, Y_{1,t}, \ldots, Y_{p,t})
\end{aligned} \qquad \text{(EQ 7)}$$

1. See table 2.2.6.
2. At first our assumption on the relative implied volatility surface seems counterproductive, but as we will see in theorem 2.4.5, if we would have started with the absolute surface, we would have to derive the relative one later.

and $\tilde{\vartheta}_t = (\tilde{\vartheta}_{0,t}, \ldots, \tilde{\vartheta}_{p,t})$ with

$$\tilde{\vartheta}_{j,t} = \sum_{i=1}^{p} \gamma_{i,j,t} \frac{\partial g}{\partial y_i}(t, Y_{1,t}, \ldots, Y_{p,t}) \quad \text{(EQ 8)}$$

We therefore have a stochastic differential equation for the relative implied volatility surface, for every **fixed moneyness** M and **fixed time to expiry** τ (see EQ 1 and EQ 6):

$$d\tilde{\sigma}_t(M, \tau) = \tilde{\eta}_t(M, \tau)dt + \tilde{\vartheta}_t(M, \tau)dW_t \quad \text{(EQ 9)}$$

where we omitted the dependency on the risk factors $Y_{i,t}$ for $i \in \{1, \ldots, p\}$.

We now have to translate the equation for the relative volatility surface (EQ 9) into an equation for the absolute one. As noted in (EQ 4, p. 44) we have $\sigma_t(\mathbf{m}^{-1}(M), t + \tau) = \tilde{\sigma}_t(M, \tau)$, which we can write as $\sigma_t(K, T) = \tilde{\sigma}_t(M, T - t)$, we therefore need (EQ 9) for changing moneyness M and time to expiry τ.

2. We first let the time to expiry change deterministicly $\tau(t) = T - t$ and keep M fixed. Itô's lemma yields, using (EQ 1) and (EQ 6):

$$d\tilde{\sigma}_t(M, T - t) = dg(t, M, T - t) = \left(\tilde{\eta}_t(M, T - t) - \frac{\partial g}{\partial \tau}(t, M, T - t)\right)dt + \tilde{\vartheta}_t(M, T - t)dW_t \quad \text{(EQ 10)}$$

3. Now we apply Itô's lemma to our moneyness function $\mathbf{m}(t, s) = \mathbf{m}(t, s, K, T, r)$, which is twice differentiable per definition 2.3.9:

$$\begin{aligned} d\mathbf{m}(t, S_t) &= \frac{\partial \mathbf{m}}{\partial t}(t, S_t)dt + \frac{\partial \mathbf{m}}{\partial s}(t, S_t)dS_t + \frac{1}{2}\frac{\partial^2 \mathbf{m}}{\partial s^2}(t, S_t)d\langle S \rangle_t \\ &= \left(\frac{\partial \mathbf{m}}{\partial t}(t, S_t) + S_t\mu_t\frac{\partial \mathbf{m}}{\partial s}(t, S_t) + \frac{1}{2}S_t^2\nu_t^2\frac{\partial^2 \mathbf{m}}{\partial s^2}(t, S_t)\right)dt + S_t\nu_t\frac{\partial \mathbf{m}}{\partial s}(t, S_t)dW_{0,t} \end{aligned} \quad \text{(EQ 11)}$$

4. Combining (EQ 10) and (EQ 11) with the help of the Itô-Venttsel formula[1] yields the desired result:

$$\begin{aligned} d\sigma_t(K, T) &= d\tilde{\sigma}_t(M_t, T - t) \\ &= \eta_t(K, T)dt + \vartheta_t(K, T)dW_t \end{aligned} \quad \text{(EQ 12)}$$

with

$$\vartheta_{0,t}(K, T) = S_t\nu_t\frac{\partial \mathbf{m}}{\partial s}(t, S_t)\frac{\partial g}{\partial M}(t, M_t, T - t) + \tilde{\vartheta}_{0,t}(M_t, T - t)$$

$$\vartheta_{i,t}(K, T) = \tilde{\vartheta}_{i,t}(M_t, T - t)$$

1. See appendix A.4.

and

$$\eta_t(K,T) = \tilde{\eta}_t(M_t, T-t) - \frac{\partial g}{\partial \tau}(t, M, T-t)$$
$$+ \frac{1}{2}\left(S_t \nu_t \frac{\partial \mathbf{m}}{\partial s}(t, S_t)\right)^2 \frac{\partial^2 g}{\partial M^2}(t, M_t, T-t)$$
$$+ \left(\frac{\partial \mathbf{m}}{\partial t}(t, S_t) + S_t \mu_t \frac{\partial \mathbf{m}}{\partial s}(t, S_t) + \frac{1}{2} S_t^2 \nu_t^2 \frac{\partial^2 \mathbf{m}}{\partial s^2}(t, S_t)\right) \frac{\partial g}{\partial M}(t, M_t, T-t)$$
$$+ S_t \nu_t \frac{\partial \mathbf{m}}{\partial s}(t, S_t) \frac{\partial u}{\partial M}(t, M_t, T-t)$$

where u is the deterministic function corresponding to $\tilde{\vartheta}_{0,t}$ and $\tilde{\vartheta}_{i,t}(M_t, T-t)$ is given in EQ 8 and $\tilde{\eta}_t(M_t, T-t)$ in EQ 7.

We can now insert (EQ 12) into (EQ 4):

$$dC_t(K,T) = \left(\Theta_t + \delta_t S_t \mu_t + \Lambda_t \eta_t(K,T) + \frac{1}{2}\Gamma_t S_t^2 \nu_t^2\right.$$
$$\left. + \frac{1}{2}\Phi_t \vartheta_t(K,T)\vartheta^t{}_t(K,T) + \Psi_t S_t \nu_t \vartheta_{0,t}(K,T)\right) dt$$
$$+ \delta_t S_t \nu_t dW_{0,t} + \Lambda_t \vartheta_t(K,T) dW_t$$

which can be written as:

$$dC_t(K,T) = \left(\Theta_t + \delta_t S_t \mu_t + \Lambda_t \eta_t(K,T) + \frac{1}{2}\Gamma_t S_t^2 \nu_t^2\right. \quad \text{(EQ 13)}$$
$$\left. + \frac{1}{2}\Phi_t \vartheta_t(K,T)\vartheta^t{}_t(K,T) + \Psi_t S_t \nu_t \vartheta_{0,t}(K,T)\right) dt$$
$$+ \hat{\vartheta}_t(K,T) dW_t$$

with $\hat{\vartheta}_{0,t}(K,T) = \delta_t S_t \nu_t + \Lambda_t \vartheta_{0,t}(K,T)$ and $\hat{\vartheta}_{i,t}(K,T) = \Lambda_t \vartheta_{i,t}(K,T)$ for $i \in \{1, \ldots, d\}$.

We hence have retrieved the desired dynamics of the call price process expressed as an Itô process.

Risk Neutral World

To derive the call option price process under the risk neutral measure and to get necessary conditions for the market price of risk process X_t we start with discounting (EQ 13).

Discounting (EQ 13) and writing this with respect to the new Brownian motion yields:

$$dC_t^*(K,T) = d\frac{C_t(K,T)}{B_t} = -rC_t^*(K,T)dt + \frac{1}{B_t}dC_t(K,T)$$
$$= \frac{1}{B_t}\Big(\Theta_t + \delta_t S_t r + \Lambda_t \eta_t(K,T) + \frac{1}{2}\Gamma_t S_t^2 \nu_t^2 \qquad \text{(EQ 14)}$$
$$+ \frac{1}{2}\Phi_t \vartheta_t(K,T)\vartheta^t{}_t(K,T) + \Psi_t S_t \nu_t \vartheta_{0,t}(K,T) - rC_t(K,T)\Big)dt$$
$$-\vartheta_t(K,T)X_t dt + \frac{1}{B_t}\hat{\vartheta}_t(K,T)dW_t^*$$

From Black-Scholes PDE (formula 2.2.4) we know that, if the stocks volatility ν is equal to the implied volatility $\sigma_t(K,T)$, we have:

$$\frac{1}{2}\sigma_t^2(K,T)S_t^2\Gamma_t = rC_t(K,T) - \Theta_t - rS_t\delta_t \qquad \text{(EQ 15)}$$

Using (EQ 15) in (EQ 14) yields:

$$dC_t^*(K,T) = \frac{1}{B_t}\Big(\Lambda_t\eta_t(K,T) + \frac{1}{2}\Gamma_t S_t^2(\nu_t^2 - \sigma_t^2(K,T)) \qquad \text{(EQ 16)}$$
$$+ \frac{1}{2}\Phi_t \vartheta_t(K,T)\vartheta^t{}_t(K,T) + \Psi_t S_t \nu_t \vartheta_{0,t}(K,T)\Big)dt$$
$$-\vartheta_t(K,T)X_t dt + \frac{1}{B_t}\hat{\vartheta}_t(K,T)dW_t^*$$

Let us pull out vega Λ_t from the first bracket in (EQ 16), and calculate the greeks gamma Γ_t, DVegaDVol Φ_t and DDeltaDVol Ψ_t relative to vega Λ_t using table 2.2.6:

$$\frac{\Gamma_t}{\Lambda_t} = \frac{1}{S_t^2(T-t)\sigma_t(K,T)}$$
$$\frac{\Phi_t}{\Lambda_t} = \frac{d_1 d_2}{\sigma_t(K,T)}$$
$$\frac{\Psi_t}{\Lambda_t} = \frac{-d_2}{S_t\sigma_t(K,T)\sqrt{T-t}}$$

where we wrote $d_i = d_i(t, S_t, K, T, r, \sigma_t(K,T))$, $i \in \{1, 2\}$ (for a definition of d_i see EQ 1, p. 34).

Inserting these formulas into (EQ 16), yields:

$$dC_t^*(K,T) = \frac{\Lambda_t}{B_t}\left(\eta_t(K,T) + \frac{1}{2}\frac{\nu_t^2 - \sigma_t^2(K,T)}{(T-t)\sigma_t(K,T)} + \frac{1}{2}\frac{d_1 d_2}{\sigma_t(K,T)}\vartheta_t(K,T)\vartheta^t{}_t(K,T)\right.$$
$$\left. - \frac{d_2}{\sigma_t(K,T)\sqrt{T-t}}\nu_t\vartheta_{0,t}(K,T)\right)dt - \vartheta_t(K,T)X_t dt + \frac{1}{B_t}\hat{\vartheta}_t(K,T)dW_t^*$$

Therefore if the process $X = (X_1, \ldots, X_d)$ should be a "Market Price of Risk Process" it has to solve the system of linear equations given in (EQ 17), for all $K \in \mathbb{R}^+$, $T \le T^*$ and $t \in [0, T]$, so that each discounted call price process $C_t^*(K,T)$ has got a drift of zero and is hence a local martingale with respect to the risk neutral measure $\mathbf{Q}$.

$$\begin{aligned}\vartheta_t(K,T)X_t = {} & \eta_t(K,T) \\ & + \frac{1}{2}\frac{\nu_t^2 - \sigma_t^2(K,T)}{(T-t)\sigma_t(K,T)} \\ & + \frac{1}{2}\frac{d_1 d_2}{\sigma_t(K,T)}\vartheta_t(K,T)\vartheta^t{}_t(K,T) \\ & - \frac{d_2}{\sigma_t(K,T)\sqrt{T-t}}\nu_t\vartheta_{0,t}(K,T)\end{aligned} \quad \text{(EQ 17)}$$

The dynamics under the equivalent measure $\mathbf{Q}$ of the implied volatility surfaces $\sigma_t(K,T)$ and $\tilde{\sigma}_t(K,T)$ can be written as (using EQ 9 and EQ 12):

$$d\tilde{\sigma}_t(M,\tau) = (\tilde{\eta}_t(M,\tau) - \tilde{\vartheta}_t(M,\tau)X_t)dt + \tilde{\vartheta}_t(M,\tau)dW_t^* \quad \text{(EQ 18)}$$

$$\begin{aligned}d\sigma_t(K,T) &= (\eta_t(K,T) - \vartheta_t(K,T)X_t)dt + \vartheta_t(K,T)dW_t^* \\ &= \eta_t^*(K,T)dt + \vartheta_t(K,T)dW_t^*\end{aligned} \quad \text{(EQ 19)}$$

with

$$\begin{aligned}\eta_t^*(K,T) = {} & -\frac{1}{2}\frac{\nu_t^2 - \sigma_t^2(K,T)}{(T-t)\sigma_t(K,T)} \\ & - \frac{1}{2}\frac{d_1 d_2}{\sigma_t(K,T)}\vartheta_t(K,T)\vartheta^t{}_t(K,T) \\ & + \frac{d_2}{\sigma_t(K,T)\sqrt{T-t}}\nu_t\vartheta_{0,t}(K,T)\end{aligned}$$

2.4.2 Stock Price Distribution

Note the possible explosion in the drift term of (EQ 19) for $t \to T$, to guarantee a solution (see footnote 3, p. 10) a necessary condition is:[1]

$$\lim_{t \to T}\left(-\frac{1}{2}\frac{\nu_t^2 - \sigma_t^2(K, T)}{(T-t)\sigma_t(K, T)} - \frac{1}{2}\frac{d_1 d_2}{\sigma_t(K, T)}\vartheta_t(K, T)\vartheta_t^t(K, T) + \frac{d_2}{\sigma_t(K, T)\sqrt{T-t}}\nu_t\vartheta_{0,t}(K, T)\right) < \infty \qquad \text{(EQ 20)}$$

for every fixed strike K and time to expiry T. If we pull out $[(T-t)\sigma_t(K, T)]^{-1}$ which goes to infinity for t to T, we see that we need:

$$\lim_{t \to T}\left(-\frac{1}{2}(\nu_t^2 - \sigma_t^2) - \frac{1}{2}(d_1 d_2)\|\vartheta_t\|^2(T-t) + d_2\nu_t\vartheta_{0,t}\sqrt{T-t}\right) = 0 \qquad \text{(EQ 21)}$$

From (EQ 1, p. 34) we immediately obtain $\lim_{t \to T}(d_1\sqrt{T-t}) = \lim_{t \to T}(d_2\sqrt{T-t}) = \frac{1}{\sigma_T}\log(S_T/K)$ using this and multiplying (EQ 21) by $2\sigma_t^2$, we can simplify (EQ 21) to:

$$\sigma_T^4 - \nu_T^2\sigma_T^2 + 2\nu_T\sigma_T M\vartheta_{0,T} - \|\vartheta_T\|^2 M^2 = 0 \qquad \text{(EQ 22)}$$

where we used $M := \log(S_T/K)$.

That is we get a polynomial, which is in the fourth degree for the implied volatility σ_T and in the second for the spot price volatility ν_T. Consequently we can solve it with respect to ν_T:

$$\nu_T = \frac{M\vartheta_{0,T}}{\sigma_T} \pm \sqrt{\sigma_T^2 + \frac{M^2}{\sigma_T^2}(\vartheta_{0,T}^2 - \|\vartheta_T\|^2)}$$

Even though it is possible to solve (EQ 22) with respect to σ_T the general solution for a polynomial of degree 4 is quite cumbersome.[2] Therefore let us assume that the implied volatility is uncorrelated to the stock price process, i.e. $\vartheta_{0,T} = 0$, then the solution for the implied volatility process is given by:

$$\sigma_T = \sqrt{\frac{\nu_T^2}{2} + \sqrt{\frac{\nu_T^4}{4} + \|\vartheta_T\|^2 M^2}} \qquad \text{(EQ 23)}$$

which shows that the implied volatility smiles around the ATM point $M = 0$, and that the volatility of volatility $\|\vartheta_T\|^2$ measures the convexity of the smile (see figure 2.4.1).

The correlation factor $\vartheta_{0,T}$ between the implied volatility and the stock price process measures where the minimum of the implied volatility smile can be found. The more strong they are positively correlated the more ITM the minimum will be (see figure 2.4.3).

1. In the following see Schönbucher [79], pp. 12-14.
2. See Bosch [14], pp. 270-273.

If we calculate the risk neutral distribution of the spot price from the implied volatility smile, using the techniques derived in chapter 2.1.7, we can see that the volatility of volatility influences the peakedness of the distribution, and therefore the size of the tails (kurtosis) (see figure 2.4.2).
The correlation between the stock price process and the volatility measures the asymmetry of the distribution (skewness), which can be observed in figure 2.4.4.

We therefore can estimate the volatility of implied volatility $\|\vartheta_T\|^2$ and the correlation between implied volatility and the stock price process $\vartheta_{0,t}$ from (observable) implied volatility surfaces.

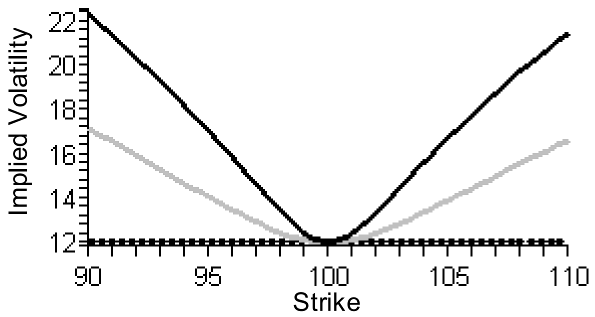


Figure 2.4.1 Implied Volatility by Volatility of Volatility

Implied Volatility as calculated by (EQ 23) with a spot price of 100, a spot price volatility ν_T of 12%, and a volatility of volatility $\|\vartheta_T\|$ of 0% (dotted), 20% (grey) and 40% (solid).

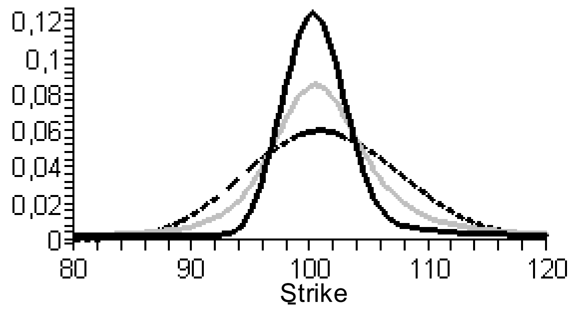


Figure 2.4.2 Risk Neutral Density by Volatility of Volatility

Risk Neutral Density of figure 2.4.1, calculated with approximation from (EQ 22, p. 27), with time to expiry 100 days, interest rate of 5% and $\varepsilon = 5$.

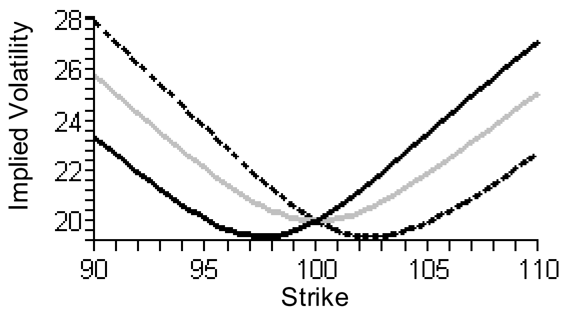


Figure 2.4.3 Implied Volatility by Correlation

Implied Volatility as calculated by (EQ 22) with a spot price of 100, a spot price volatility ν_T of 20%, a volatility of volatility $\|\vartheta_T\|$ of 40% and a correlation $\vartheta_{0,T}$ of -10% (dotted), 0% (grey) and 10% (solid). We solved (EQ 22) with respect to σ_T with the help of a computer program.

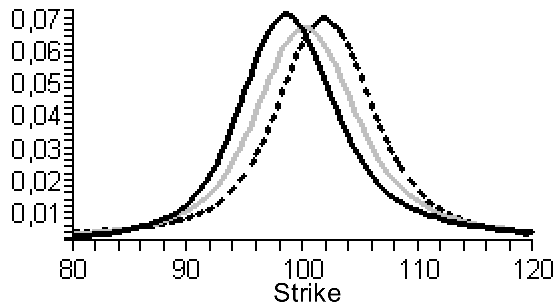


Figure 2.4.4 Risk Neutral Density by Correlation

Risk Neutral Density of figure 2.4.3, calculated with approximation from (EQ 22, p. 27), with time to expiry 100 days, interest rate of 5% and $\varepsilon = 5$.

2.4.3 Arbitrage Free and Complete Market

In the stochastic implied volatility model it is not obvious that an equivalent martingale measure can be found or if this is unique:

- If we have infinitely many traded underlyings (EQ 17) can only have a solution $X = (X_1, \ldots, X_d)$ if we can define X independently of K and T.
- Even if we can find a solution to (EQ 17) it is not necessary true, that we can derive an equivalent martingale measure. As we have seen in chapter 2.1, the existence of a solution to (EQ 17) does not necessary lead to an equivalent martingale measure, we also need some integrability properties, like Novikov's condition.
- If we have an equivalent martingale measure for the call options $C_t(K_1, T_1), \ldots, C_t(K_p, T_p)$ we know from chapter 2.1 that the equivalent martingale measure would be unique if and only if the $d \times d$-matrix $(\vartheta_{j,t}(K_i, T_i))_{ij}$ is injective.

We will assume that the market satisfies NFLVR and completeness, hence we have a unique equivalent martingale measure.[1] However, the model is free from static arbitrage opportunities, by construction.

2.4.4 Martingale Pricing

Arbitrage arguments are forcing a strong connection between the stock price volatility process ν_t and the implied volatility process $\sigma_t(K, T)$.[2] Let us assume for example, that we are in the Black-Scholes framework with constant implied volatility $\sigma_t(K, T) = c$, it would immediately follow from (EQ 17) that $\nu_t = c$ must be satisfied, because from (EQ 12) we know $\eta_t(K, T) = 0$ and $\vartheta_t(K, T) = 0$. Another example, let σ_t be deterministic and just dependent on the time t, i.e. $\vartheta_t(K, T) = 0$, in this case (EQ 17) simplifies to

$$0 = \eta_t(K, T) + \frac{1}{2}\frac{\nu_t^2 - \sigma_t^2(K, T)}{(T-t)\sigma_t(K, T)} \qquad \text{(EQ 24)}$$

we already know[3] that the correct implied volatility is the root-mean-square volatility $\sigma_t(K, T) = \sqrt{\frac{1}{T-t}\int_t^T \nu_s^2 ds}$, which implies (differentiating with respect to the time t):

$$d\sigma_t(K, T) = \frac{1}{2\sigma_t(K, T)}\frac{1}{T-t}(\sigma_t^2(K, T) - \nu_t^2)dt$$

which exactly defines the process $\eta_t(K, T)$ from (EQ 24).[4]

1. See also Hafner [51], pp. 68-70.
2. In fact under the assumption of NFLVR, we know that the market model must have an equivalent martingale measure (theorem 2.1.24), because all randomness is induced by Brownian Motion this must come from a Girsanov transformation (theorem 2.1.30), i.e. (EQ 17) must have got a solution. It is common to write "no-arbitrage arguments induce...", even though in fact the stronger assumption of NFLVR is normally meant.
3. As seen on page 48.
4. Similar examples can be found in Schönbucher [79], pp. 11-12.

This connection between implied and realized volatility is the main result of the stochastic implied volatility model, i.e. that the volatility process of the stock ν_t can be written as a process of the implied volatility. A connection between the implied volatility and the stock price volatility is crucial because otherwise, the process ν_t would have to be modelled like in standard stochastic volatility models and the stochastic implied volatility model could not be complete.

Theorem 2.4.5 Implied Volatility Converges to Stock Price Volatility
We have

$$\nu_t = \lim_{T \to t} \tilde{o}(\mathbf{m}(t, S_t, S_t e^{r(T-t)}, T, r), T-t) = \tilde{o}(\mathbf{m}(t, S_t, S_t, t, r), 0)$$

Proof:[1] From theorem 2.2.2 we know the Black-Scholes price process of an ATM European call option $C(t, S_t;K)$, i.e. having a strike $K := S_t e^{r(T-t)}$, is given by

$$C_{BS}(t, S_t;S_t e^{r(T-t)}) = S_t(\mathbf{N}(d_1) - \mathbf{N}(d_2)) \quad \text{(EQ 25)}$$

with $d_1(t, S_t;K) = \frac{1}{2}\tilde{o}(\mathbf{m}(t, S_t, S_t e^{r(T-t)}, T, r), T-t)\sqrt{T-t}$ and $d_2(t, S_t;K) = -d_1(t, S_t;K)$.

Using a Tylor approximation of the cumulative normal distribution $\mathbf{N}$ around 0 yields:

$$\begin{aligned}\mathbf{N}(d_1(t, S_t;S_t e^{r(T-t)})) &= \frac{1}{2} + \frac{1}{\sqrt{2\pi}} d_1(t, S_t;S_t e^{r(T-t)}) + R(d_1(t, S_t;S_t e^{r(T-t)})) \\ &= \frac{1}{2} + \frac{\tilde{o}(\mathbf{m}(t, S_t, S_t e^{rT-t}, T, r), T-t)\sqrt{T-t}}{2\sqrt{2\pi}} + R(d_1(t, S_t;S_t e^{r(T-t)}))\end{aligned} \quad \text{(EQ 26)}$$

and

$$\mathbf{N}(d_2(t, S_t;S_t e^{r(T-t)})) = \frac{1}{2} - \frac{\tilde{o}(\mathbf{m}(t, S_t, S_t e^{rT-t}, T, r), T-t)\sqrt{T-t}}{2\sqrt{2\pi}} - R(d_1(t, S_t;S_t e^{rT-t})) \quad \text{(EQ 27)}$$

with a rest term $R(x)$ satisfying $\lim_{x \to 0} R(x) = 0$.
Inserting (EQ 26) and (EQ 27) into (EQ 25) gives us:

$$C_{BS}(t, S_t;S_t e^{r(T-t)}) = S_t \frac{\tilde{o}(\mathbf{m}(t, S_t, S_t e^{rT-t}, T, r), T-t)\sqrt{T-t}}{\sqrt{2\pi}} + 2S_t R(d_1(t, S_t;S_t e^{r(T-t)})) \quad \text{(EQ 28)}$$

1. See Hafner [51], pp. 70-72.

Rearranging (EQ 28) and using $\lim_{T \to t} R(d_1(t, S_t; S_t e^{r(T-t)})) = 0$ gives us a term for the implied volatility:

$$\tilde{\sigma}(\mathbf{m}(t, S_t, S_t, T, r), 0) = \lim_{T \to t} \tilde{\sigma}(\mathbf{m}(t, S_t, S_t e^{rT-t}, T, r), T-t)$$
$$= \lim_{T \to t} \frac{\sqrt{2\pi}}{\sqrt{T-t}} \frac{C_{BS}(t, S_t; S_t e^{r(T-t)})}{S_t} \qquad \text{(EQ 29)}$$

To prove the statement we have to show, that $\lim_{T \to t} \frac{\sqrt{2\pi}}{\sqrt{T-t}} \frac{C_{BS}(t, S_t; S_t e^{r(T-t)})}{S_t}$ is equal to the stock price volatility ν_t.

From theorem 2.1.34 we know that we can derive the Black-Scholes price by

$$C_{BS}(t, S_t; S_t e^{r(T-t)}) = e^{-r(T-t)} \mathbb{E}_Q[(S_T - S_t e^{r(T-t)})^+ \mid \mathcal{F}_t]$$

Because we consider the expiry T close to t, we can approximate

$$S_T \approx S_t e^{r(T-t)} + S_t \nu_t (W_T^* - W_t^*)$$

We now have:

$$\begin{aligned}
&\lim_{T \to t} \frac{1}{\sqrt{T-t}} C_{BS}(t, S_t; S_t e^{r(T-t)}) \\
&= \lim_{T \to t} \frac{1}{\sqrt{T-t}} e^{-r(T-t)} \mathbb{E}_Q[(S_T - S_t e^{r(T-t)})^+ \mid \mathcal{F}_t] \\
&= \lim_{T \to t} \frac{1}{\sqrt{T-t}} e^{-r(T-t)} S_t \nu_t \mathbb{E}_Q[(W_T^* - W_t^*)^+ \mid \mathcal{F}_t] \\
&= \lim_{T \to t} \frac{1}{\sqrt{T-t}} e^{-r(T-t)} S_t \nu_t \mathbb{E}_Q[(W_{T-t}^* - W_0^*)^+] \\
&= \lim_{T \to t} \frac{1}{\sqrt{T-t}} e^{-r(T-t)} S_t \nu_t \frac{\sqrt{T-t}}{\sqrt{2\pi}} \qquad \text{(EQ 30)} \\
&= \frac{1}{\sqrt{2\pi}} S_t \nu_t
\end{aligned}$$

In (EQ 30) we used the fact that for a normal distributed random variable Z with mean 0 and variance ν, we have $\mathbb{E}[(Z)^+] = \frac{\sqrt{\nu}}{\sqrt{2\pi}}$.

If we now insert this result into (EQ 29) the proof is finished. q.e.d.

We can now apply Theorem 2.1.34, “Risk-Neutral Asset Pricing“

Theorem 2.4.6 Risk-Neutral Asset Pricing

Let H be any contingent claim with maturity date T. We have:

$$\forall t \in [0, T] \qquad \Pi_t(H) = S_{0,t} \mathbb{E}_Q\left[\frac{H}{S_{0,T}} \mid \mathcal{F}_t\right]$$

the expectation is calculated under the joint evolution of the stock price under the risk-neutral measure $\mathbf{Q}$:

$$dS_t = rS_t dt + \nu_t S_t dW^*_{0,t} \qquad \text{(EQ 31)}$$

and its volatility

$$d\nu_t = (\tilde{\eta}_t(\mathbf{m}(t, S_t, S_t, t, r), 0) - \tilde{\vartheta}_t(\mathbf{m}(t, S_t, S_t, t, r), 0)X_t)dt + \tilde{\vartheta}_t(\mathbf{m}(t, S_t, S_t, t, r), 0)dW_t{}^* \qquad \text{(EQ 32)}$$

where $\tilde{\eta}_t$ is defined in (EQ 7) and $\tilde{\vartheta}_t$ in (EQ 8)

Proof: Immediately follows from theorem 2.1.34. See Hafner [51] p. 72. q.e.d.

2.4.5 Model Applications

Obviously to apply this model, we first have to find a reasonable number of risk factors to model the relative volatility surface $\tilde{\sigma}(M, \tau)$ and then find an Itô process for each risk factor (see assumption iv, p. 54).

In the following we will use the log-simple moneyness (definition 2.3.11) as our moneyness function $\mathbf{m}$:[1]

$$\mathbf{m}(t, s, K, T, r) = \log\left(\frac{K}{se^{rT-t}}\right) \qquad \text{(EQ 33)}$$

Real World Model

Hafner [51] analysed the implied volatility surface from DAX options and came to the conclusion to consider four risk factors, called (Y_1, Y_2, Y_3, Y_4), to model the function describing the relative implied volatility surface $g(M, \tau, y)$ in the following way:[2]

$$g(t, M, \tau, y_1, y_2, y_3, y_4) = \exp(y_1) + y_2 M(1 + \rho_1 \log(1+\tau)) + y_3 M^2(1 + \rho_2 \log(1+\tau)) + y_4 \log(1+\tau)$$

1. For details see Hafner [51], pp. 85-87.
2. Hafner [51], pp. 87-90.

with[1]

$$\rho_1 = -1.6977 \quad \text{(EQ 34)}$$

$$\rho_2 = -3.3768 \quad \text{(EQ 35)}$$

The relative implied volatility surface is hence given by:

$$\begin{aligned} \tilde{\sigma}_t(M, \tau) &= g(t, M, \tau, Y_{1,t}, Y_{2,t}, Y_{3,t}, Y_{4,t}) \\ &= \exp(Y_{1,t}) + Y_{2,t}M(1 + \rho_1 \log(1+\tau)) \\ &\quad + Y_{3,t}M^2(1 + \rho_2 \log(1+\tau)) + Y_{4,t}\log(1+\tau) \end{aligned} \quad \text{(EQ 36)}$$

with the following interpretation:[2]

- Y_1 measures the general level of implied volatility.
- Y_2 resembles the slope of the implied volatility surface.
- Y_3 defines the curvature of the implied volatility surface.
- Y_4 measures the term-structure of the implied volatility surface.

Further on he derives that each risk factor can be modelled by a mean-reverting Ornstein-Uhlenbeck process.[3]

$$dY_{i,t} = a_i(c_i - Y_{i,t})dt + \sum_{j=0}^{4} \gamma_{i,j} dW_{j,t} \quad \text{(EQ 37)}$$

We already proved that the stock price volatility ν_t is given by $\tilde{\sigma}(\mathbf{m}(t, S_t, S_t, t, r), 0)$ (theorem 2.4.5). Using the moneyness function defined in (EQ 33) and using (EQ 36) we gain:

$$\nu_t = \tilde{\sigma}(t, 0, 0) = \exp(Y_{1,t}) \quad \text{(EQ 38)}$$

And hence:

$$dS_t = \mu S_t dt + \exp(Y_{1,t}) S_t dW_{0,t} \quad \text{(EQ 39)}$$

Estimations from data between 1995 and 2002 gives:

Table 2.4.7 Risk factor estimates[a]

Risk factor	a_i	c_i	$\gamma_{i,0}$	$\gamma_{i,1}$	$\gamma_{i,2}$	$\gamma_{i,3}$	$\gamma_{i,4}$
Y_1	2.7575	-1.4797	-0.6155	0.7889	0	0	0

1. Hafner [51], p.92.
2. See Hafner [51], pp. 94-96.
3. See Hafner [51], pp. 115-116.

Table 2.4.7 Risk factor estimates[a]

Risk factor	a_i	c_i	$\gamma_{i,0}$	$\gamma_{i,1}$	$\gamma_{i,2}$	$\gamma_{i,3}$	$\gamma_{i,4}$
Y_2	7.9951	-0.5013	-0.1009	-0.0365	0.5543	0	0
Y_3	57.3609	1.4601	0.3294	-1.9553	2.1043	10.0484	0
Y_4	14.0119	0.0144	0.2522	-0.5476	0.0303	-0.0682	0.40790

a. See Hafner [51], pp. 123-124.

Formula 2.4.8 Real-World Model (Hafner)

$$\begin{aligned} dS_t &= \mu S_t dt + \nu_t S_t dW_{0,t} \\ \nu_t &= \exp(Y_{1,t}) \\ dY_{1,t} &= -2.8(1.5 + Y_{1,t})dt - 0.6dW_{0,t} + 0.8dW_{1,t} \end{aligned} \quad \text{(EQ 40)}$$

If we let the time t go to infinity we derive the invariant distribution of a mean-reverting Ornstein-Uhlenbeck process, which in our case for $Y_{1,t}$ is given by a normal distribution with mean -1.5 and variance of $1/5.6$.[1]

Risk Neutral Model

Let us now state the model under the risk neutral measure $\mathbf{Q}$. Recall that under the risk-neutral measure $\mathbf{Q}$ the model is given by (EQ 31) and (EQ 32):

$$\begin{aligned} dS_t &= rS_t dt + \nu_t S_t dW^*_{0,t} \\ d\nu_t &= (\tilde{\eta}_t(\mathbf{m}(t, S_t, S_t, t, r), 0) - \tilde{\vartheta}_t(\mathbf{m}(t, S_t, S_t, t, r), 0)X_t)dt + \tilde{\vartheta}_t(\mathbf{m}(t, S_t, S_t, t, r), 0)dW_t^* \end{aligned}$$

where X_t is the market price of risk process, $\tilde{\eta}_t$ and $\tilde{\vartheta}_t$ are defined in (EQ 7) and (EQ 8), respectively:

$$\begin{aligned} \tilde{\eta}_t &= \frac{\partial g}{\partial t}(t, Y_{1,t}, \dots, Y_{p,t}) + \sum_{i=1}^{p} \alpha_{i,t} \frac{\partial g}{\partial y_i}(t, Y_{1,t}, \dots, Y_{p,t}) \\ &+ \frac{1}{2} \sum_{i=1}^{p} \sum_{k=1}^{p} \sum_{j=0}^{p} \gamma_{i,j,t} \gamma_{k,j,t} \frac{\partial^2 g}{\partial y_i \partial y_k}(t, Y_{1,t}, \dots, Y_{p,t}) \end{aligned} \quad \text{(EQ 41)}$$

and $\tilde{\vartheta}_t = (\tilde{\vartheta}_{0,t}, \dots, \tilde{\vartheta}_{p,t})$ with

$$\tilde{\vartheta}_{j,t} = \sum_{i=1}^{p} \gamma_{i,j,t} \frac{\partial g}{\partial y_i}(t, Y_{1,t}, \dots, Y_{p,t}) \quad \text{(EQ 42)}$$

1. Fouque et al. [46], p. 41.

In our estimated model we can simplify (EQ 41) to:

$$\begin{aligned}\tilde{\eta}_t(M,\tau) &= \alpha_{1,t}\exp(Y_{1,t}) + \alpha_{2,t}M(1+\rho_1\log(1+\tau)) \\ &\quad + \alpha_{3,t}M^2(1+\rho_2\log(1+\tau)) + \alpha_{4,t}\log(1+\tau) + \frac{1}{2}\sum_{j=0}^{4}\gamma_{1,j}^2\exp(Y_{1,t})\end{aligned} \quad \text{(EQ 43)}$$

and (EQ 42) to:

$$\begin{aligned}\tilde{\vartheta}_{j,t}(M,\tau) &= \gamma_{1,j}\exp(Y_{1,t}) + \gamma_{2,j}M(1+\rho_1\log(1+\tau)) \\ &\quad + \gamma_{3,j}M^2(1+\rho_2\log(1+\tau)) + \gamma_{4,j}\log(1+\tau)\end{aligned} \quad \text{(EQ 44)}$$

where we have $\alpha_{i,t} = a_i(c_i - Y_{i,t})$, compare (EQ 2) and (EQ 37), a_i, c_i, $\gamma_{i,j}$ given in table 2.4.7, ρ_1 in (EQ 34) and ρ_2 in (EQ 35).

Since $m(t, S_t, S_t, t, r)$ is zero, we can write the volatility process as:

$$\begin{aligned}dv_t &= (\tilde{\eta}_t(0,0) - \tilde{\vartheta}_t(0,0)X_t)dt + \tilde{\vartheta}_t(0,0)dW_t^* \\ &= \left(\alpha_{1,t}\exp(Y_{1,t}) + \frac{1}{2}\sum_{j=0}^{4}\gamma_{1,j}^2\exp(Y_{1,t}) - \sum_{i=1}^{4}\gamma_{1,j}\exp(Y_{1,t})\cdot X_{j,t}\right)dt \\ &\quad + \sum_{j=0}^{4}\gamma_{1,j}\exp(Y_{1,t})dW_{j,t}^*\end{aligned}$$

Which can further be simplified using $\gamma_{1,j} = 0$ for $j \in \{2,3,4\}$ and $v_t = \exp(Y_{1,t})$ to:

$$\begin{aligned}dv_t &= v_t\left(a_1(c_1 - \ln(v_t)) + \frac{1}{2}(\gamma_{1,0}^2 + \gamma_{1,1}^2) - (\gamma_{1,0}X_{0,t} + \gamma_{1,1}X_{1,t})\right)dt \\ &\quad + v_t\gamma_{1,0}dW_{0,t}^* + v_t\gamma_{1,1}dW_{1,t}^*\end{aligned}$$

Note that $X_{0,t} = (\mu_t - r)/v_t$ was already derived in (EQ 3). At first it appears as if the risk factors $Y_{j,t}$ for $j \in \{2,3,4\}$ do not have any impact on the volatility. However, the risk factors under the measure **Q** are given by:

$$dY_{i,t} = \left(a_i(c_i - Y_{i,t}) - \sum_{j=1}^{4}\gamma_{i,j}X_{j,t}\right)dt + \sum_{j=0}^{4}\gamma_{i,j}dW_{j,t}^* \quad \text{(EQ 45)}$$

Hence $X_{1,t}$, and therefore also v_t, in general depends on all volatility risk factors.[1]

As pointed out by Hafner this derived model has one major flaw. The problem is that the necessary condition (EQ 20) for an arbitrage free market is not satisfied in our model.[2] One way to deal with this

1. See Hafner [51], p. 133.
2. See Hafner [51], pp. 133-137.

is to only consider a specific set of expiry times and moneyness values to get an equivalent martingale measure that at least excludes arbitrage opportunities locally. However, in this case we have more than one possible equivalent martingale measure and hence the market model cannot be complete anymore.[1]

Before we can actually price contingent claims one first has to derive the dynamics of the market price of volatility risk processes $(X_{1,t}, X_{2,t}, X_{3,t}, X_{4,t})$ and then calibrate these to the market.

The result of Hafner, analysing four different time to maturities of 10, 40, 70 and 100 days and moneyness values between -0.05 and 0.05, shows that $(X_{1,t}, X_{2,t}, X_{3,t}, X_{4,t})$ can be modelled by:[2]

$$\begin{aligned} X_{1,t} &= \kappa_1 + \kappa_2 Y_{1,t} + \kappa_3 Y_{1,t}^2 + \kappa_4 Y_{1,t} Y_{4,t} + \kappa_5 Y_{2,t} Y_{3,t} \\ X_{2,t} &= \kappa_6 \\ X_{3,t} &= \kappa_7 \\ X_{4,t} &= \kappa_8 \end{aligned} \qquad \text{(EQ 46)}$$

with $\kappa_j \in \mathbb{R}$ for $j \in \{1, \ldots, 8\}$ and $Y_{i,t}$ defined in (EQ 45) for $i \in \{1, 2, 3, 4\}$.

Data estimation on 30 December 2002, gives an estimate for κ of:[3]

$$\begin{aligned} \kappa = (&-8.1937, -8.6625, -0.1517, 31.5799, -1.2344, \\ &-39.4111, -3.6882, -8.1937, -0.7633) \end{aligned} \qquad \text{(EQ 47)}$$

Putting everything together yields the following model:

Formula 2.4.9 Risk-Neutral Model (Hafner)

$$\begin{aligned} dS_t &= rS_t dt + \nu_t S_t dW_{0,t}^* \\ d\nu_t &= \nu_t\left(a_1(c_1 - \ln(\nu_t)) + \frac{1}{2}(\gamma_{1,0}^2 + \gamma_{1,1}^2) - (\gamma_{1,0}X_{0,t} + \gamma_{1,1}X_{1,t})\right)dt \\ &\quad + \nu_t\gamma_{1,0}dW_{0,t}^* + \nu_t\gamma_{1,1}dW_{1,t}^* \end{aligned} \qquad \text{(EQ 48)}$$

Theorem 2.4.10 Risk-Neutral Asset Pricing[4]

If $\mathbf{Q}$ is defined by the "Market Price of Risk Process" in (EQ 46) and $X_{0,t} = (\mu - r)/\nu_t$.

Then the price process for every European contingent claim H with maturity T is given by:

$$\Pi_t(H) = \exp(-r(T-t))\mathbb{E}_Q[H \mid \mathcal{F}_t]$$

where the stock price and volatility process under $\mathbf{Q}$ are given in (EQ 48).

We could now use standard Monte Carlo Methods to calculate the price process for exotic options, examples are given in Hafner [51], pp. 145-158.

1. See Hafner [51], 135.
2. See Hafner [51], pp. 135-137.
3. See Hafner [51], p. 142.
4. Hafner [51], p.137.

CHAPTER 2.5 Market Models with Frictions

As noted in the introduction, the second part of this thesis analysis the cross sectional distribution of bid/ask prices, obviously our previous analysis cannot easily be generalised to allow for differences in bid/ask prices. For example an arbitrary small spread between the bid and the ask price would lead to infinite transaction costs for a continuously hedged contingent claim[1] and hence no bounds for the replicated contingent claim are implied, i.e. we do not have an analogue to theorem 2.1.33. One suggesting way would be to only consider discrete revision intervals Δt, but then either the hedging error will be significant (Δt big) or the transaction costs will be too expansive (Δt small), as noted by Leland [63], pages 1283-1284.

As was shown by Leland [63], one way to handle this is in the Black-Scholes framework is to modify the volatility parameter:

$$\bar{\nu} = \nu\sqrt{1 + k/(\nu\sqrt{\Delta t}) \cdot \sqrt{2/\pi}} \qquad \text{(EQ 1)}$$

where k is the proportional cost of a round trip, Δt the revision interval and ν the underlying volatility. Leland proved that if we let the revision interval get smaller and smaller ($\Delta t \to 0$), we have that the trading strategy $\phi = (C - \delta S, \delta)$ has a value of $(S-K)^+$ at expiry, inclusive of transaction costs.[2] With C, the price of a call option with strike K, and Black-Scholes delta δ being priced with the modified volatility from (EQ 1).
Pricing the option once with the volatility parameter given in (EQ 1) and once with (EQ 2) yields an upper and a lower bound on the option price. Knowing all his fees and costs a market maker can deduce a minimal bid/ask spread in this way. Inside this minimal spread the market maker would loose money in a competitive market.[3]

$$\underline{\nu} = \nu\sqrt{1 - k/(\nu\sqrt{\Delta t}) \cdot \sqrt{2/\pi}} \qquad \text{(EQ 2)}$$

The result of Leland can be generalized to convex payoff functions and also to European contingent claims, with an arbitrary payoff function.[4] However, one striking problem occurs, if the Leland number $A := k/(\nu\sqrt{\Delta t}) \cdot \sqrt{2/\pi}$ (following a definition from Avellaneda and Parás [5], p. 65) is higher than one. In 1994 Avellaneda and Parás [5] considered super-replicating strategies, which are strategies that have a value higher or equal to the payoff from the contingent claim at expiry, to derive results for $A \geq 1$. These approaches generally result into closer bounds, because as pointed out by

1. The stock has to have infinite variation in a continuous time model, see chapter 2.1.8.
2. Leland [63] pp. 1290-1292.
3. See Leland [63] pp. 1300.
4. See Hoggard et al. [54].

Bensaïd et al. [7], in a discrete-time framework with transaction costs it can be cheaper to dominate, rather than replicate, the payoff at expiry. Monoyios [67] considered exponential utility in a binomial tree model with proportional transaction costs to gain (in general) closer bounds, than derived by the procedure developed by Leland. A similar approach can be found in Perrakis and Lefoll [71] to derive bounds on an American call with dividends. However, as proved by Cvitanic et al. [26] in a general time-continuous model (see chapter 2.1.2)[1], the least expensive super-replicating strategy for a European path-independent contingent claim is the trivial buy-and-hold strategy.[2]

So far we always measured wealth in the riskless bond, i.e. at expiry we always assumed that the holdings in the other assets are converted into our riskless bond. As pointed out by Kabanov [59] this becomes especially peculiar in a framework with several traded currency units and therefore proposed a numeraire-free model with vector-valued portfolios. These techniques also lead to a very different approach to pricing contingent claims in a non-frictionless environment. Using these techniques Schachermayer [77] first derived a version of the First Fundamental Theorem of Asset Pricing (theorem 2.1.28) and then Campi and Schachermayer [17] proved a counterpart of the Risk Neutral Asset Pricing formula (theorem 2.1.34) in an arbitrage free model, with proportional transaction costs.

2.5.1 Model Assumptions

We will consider the Black-Scholes market model, defined in chapter 2.2, with a symmetric bid and ask price process around the mid-price process S_t:

$$S_{t,\,bid} := S_t(1 - k/2) \qquad \text{(EQ 3)}$$

$$S_{t,\,ask} := S_t(1 + k/2) \qquad \text{(EQ 4)}$$

where k represents the proportional cost of a round trip.
Further we assume that the riskless interest rate for lending and borrowing money is the same.

Following Avellaneda and Parás [5] we can summarize the results of Leland [63] and Hoggard et al. [54] in the following lemma:

Lemma 2.5.1 Pricing PDE With Transaction Costs

Let us write Δt for the constant revision interval. Assume that (EQ 5) has a solution $P(t, S)$ with $P : [0, T] \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ in $\mathcal{C}^{1,2}$ (so that Itô's formula can be applied).

$$rP(t, S_t) = \frac{\partial P}{\partial t} + \frac{\tilde{\nu}^2(\Gamma)}{2} S_t^2 \frac{\partial^2 P}{\partial S^2} + rS_t \frac{\partial P}{\partial S} \qquad \text{(EQ 5)}$$

1. With perfect correlation between the stock price process and the volatility and drift process. And two technical conditions have to be applied to the volatility process ν_t, see Cvitanic et al. [26], p. 37.
2. Surprisingly as a by-product Cvitanic et al. [26] got that the optimal super-replicating strategy in a frictionless stochastic volatility model with unbounded volatility process is also the trivial buy-and-hold strategy. As already mentioned in chapter 2.3.6.

with $\tilde{\nu}^2(\Gamma) = \nu^2(1 + A \cdot \mathrm{sign}(\Gamma))$, $A := k/(\nu\sqrt{\Delta t}) \cdot \sqrt{2/\pi}$ the Leland number, $\Gamma = \frac{\partial^2 P}{\partial S^2}$ and final condition $P(T, S_T) = H(S_T)$.

Then the trading strategy defined by $\phi_t^1 = \frac{\partial}{\partial S}P(t, S_t)$ and $\phi_t^0 = P(t, S_t) - \phi_t^1 S_t$ replicates the payoff $H(S_T)$ at expiry up to an approximation error of order $(T - t) \cdot \nu \cdot \sqrt{\Delta t} \cdot S_t$.

Proof: See Avellaneda and Parás [5], pp. 68-71. q.e.d.

Note that if $P(t, S)$ is either convex or concave for all S the SDE (EQ 5) simplifies to the Black-Scholes PDE (formula 2.2.4) and has a solution in the convex case and for small A in the concave case.[1]

2.5.2 *No-Arbitrage Pricing*

Let us apply lemma 2.5.1 to derive bounds on a European call option

Figure 2.5.2 Leland Bid/Ask Spread

Bid and ask price derived by (EQ 1) and (EQ 2), with a volatility $\nu = 0.2$, spot $S = 6000$, interest rate $r = 0.05$, round trip cost $k = 0.0002$ and revision interval of $\Delta t = 0.002$.

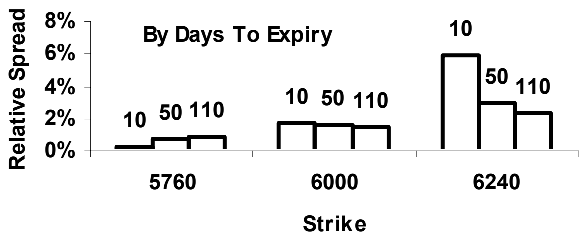


Figure 2.5.3 Leland Bid/Ask Relative Spread

Bid and ask price as in figure 2.5.2 divided by unmodified Black-Scholes call price.

It strikes out that the derived (absolute) spread is highest for near ATM options and increasing with time to expiry (see figure 2.5.2). In figure 2.5.3 we see that the relative spread is highest for short-term OTM options. Another feature of Leland’s derived spread is the asymmetry around the (unmodified) Black-Scholes price. Only for very strong ITM or OTM options the fair option price is closer to the bid price than to the ask price (figure 2.5.4). However, for more realistic round trip costs, the bid and ask price is almost symmetric (figure 2.5.5). Note that liquid futures on the DAX normally have a round trip cost, i.e. two times the spread, of two tick sizes, we assumed a spread of one index point to also compensate other costs. Using a mid-price of 6000, we hence derive a proportional round trip cost of (see EQ 3 and EQ 4):

$$k = \frac{S_{ask} - S_{bid}}{S} = \frac{1}{6000} \approx 0.0002$$

1. Avellaneda and Parás [5], p. 71-74.

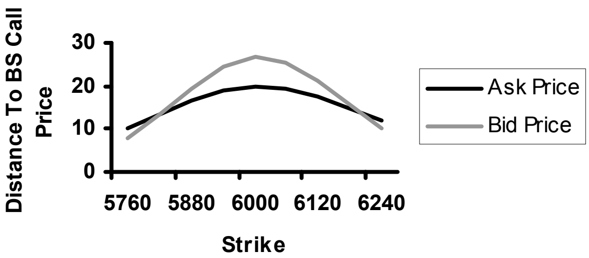


Figure 2.5.4 Leland Bid/Ask Spread

Bid and ask price derived by (EQ 1) and (EQ 2), with a volatility $\nu = 0.2$, time to expiry $T = 10$ days, spot $S = 6000$, interest rate $r = 0.05$, round trip cost $k = 0.02$ and revision interval of $\Delta t = 0.02$.

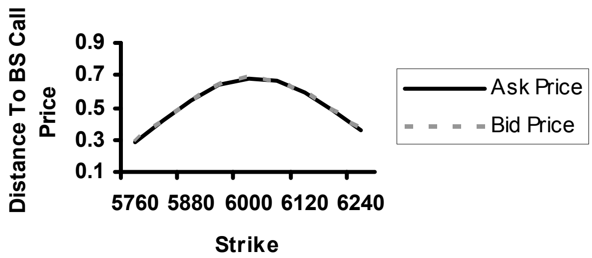


Figure 2.5.5 Leland Bid/Ask Spread

Bid and ask price derived by (EQ 1) and (EQ 2), with a volatility $\nu = 0.2$, time to expiry $T = 10$ days, spot $S = 6000$, interest rate $r = 0.05$, round trip cost $k = 0.0002$ and revision interval of $\Delta t = 0.002$.

2.5.3 Application of No-Arbitrage

We will point out the differences induced by transaction costs by reanalyzing put-call parity (lemma 2.1.36). Already in 1974 Gould and Galai [49] derived a similar result.

Lemma 2.5.6 Put-Call Parity with Frictions[1]

Let $C_{t,\,ask}$ and $C_{t,\,bid}$ denote the ask and bid price of a European call option, $P_{t,\,ask}$ and $P_{t,\,bid}$ the ask and bid price of a European put, $S_{t,\,ask}$ and $S_{t,\,bid}$ the ask and bid price of the underlying stock, respectively. Let β denote the lending and α the borrowing interest rate.

1. We have the following inequalities:

$$C_{t,\,ask} - P_{t,\,bid} \geq S_{t,\,bid} - K \cdot \exp(-\beta(T-t)) \qquad \text{(EQ 6)}$$

$$P_{t,\,ask} - C_{t,\,bid} \geq K \cdot \exp(-\alpha(T-t)) - S_{t,\,ask} \qquad \text{(EQ 7)}$$

2. Further on, equality in (EQ 6) and (EQ 7) holds if and only if bid prices are equal to ask prices.

Proof:

1.) "(EQ 6)": If $C_{t,\,ask} - P_{t,\,bid} < S_{t,\,bid} - K \cdot \exp(-\beta(T-t))$ we buy a call, and sell a put and a stock, which will cost us less than $-K\exp(-\beta(T-t))$, hence we can put $x > K\exp(-\beta(T-t))$ in a bank account. At expiry we have to pay $(S_{T,\,bid} - K) - S_{T,\,bid} = -K$, but our bank account can compensate this loss, because we have more than K in it. We therefore would make a guaranteed profit., which proves (EQ 6).

"(EQ 7)": If $P_{t,\,ask} - C_{t,\,bid} < K \cdot \exp(-\alpha(T-t)) - S_{t,\,ask}$ we sell a call, buy a put and a stock, for that we will borrow $x < K \cdot \exp(-\alpha(T-t))$ money. At expiry we therefore get

1. (EQ 6) and (EQ 7) directly follow from a more general version used by Ackert and Tian [1], p. 1613.

$(K - S_{T,\,ask}) + S_{T,\,ask} = K$ which is more than the loan at expiry in our bank account, granting us a safe profit.

2.) " $\Leftarrow$ ": If bid is equal to ask price, equality in (EQ 6) immediately follows if we multiply (EQ 7) with -1.

" $\Rightarrow$ ": On the other hand if we have equality in (EQ 6) and (EQ 7), we have

$$(C_{t,\,ask} - C_{t,\,bid}) + (P_{t,\,ask} - P_{t,\,bid}) + (S_{t,\,ask} - S_{t,\,bid}) + K(e^{-\beta(T-t)} - e^{-\alpha(T-t)}) = 0 \qquad \text{(EQ 8)}$$

Because the ask price must be higher than the bid price and $\alpha \geq \beta$ must be satisfied, we have only non-negative terms in the sum (EQ 8), which implies that they have to be 0, i.e. the bid price must be equal to the ask price. q.e.d.

Note that, if $S_{t,\,ask} > S_{t,\,bid}$, we cannot substitute $S_{t,\,bid}$ with $S_{t,\,ask}$ in (EQ 6), because the payoff from a long call and a short put is given by:

$$\begin{aligned}(S_{T,\,bid} - K)^+ - (K - S_{T,\,ask})^+ &= (S_T - (K + \delta))^+ - ((K + \delta) - S_T)^+ \\ &= S_T - (K + \delta) \\ &= S_{T,\,bid} - K\end{aligned}$$

where we assumed that the spread 2δ is symmetric around the mid-price $S_T := (S_{T,\,ask} + S_{T,\,bid})/2$.

CHAPTER 2.6 Summary

In this chapter we saw that the necessary condition of an arbitrage free market is not sufficient to derive an equivalent martingale measure and hence a martingale pricing formula in a continuous-time framework. Instead we had to consider a stronger assumption as was first done by Delbaen and Schachermayer [27], namely "no free lunch with vanishing risk", which is still meaningful to assume from a modelling point of view. To establish if a market model is arbitrage free or complete one often ignores integrability problems, leading to only consider solutions of a linear equation system. In this context we proved – following Björk [10] – that a surjective volatility matrix is sufficient for the existence of a martingale measure. And that a surjective, transposed volatility matrix is sufficient for a complete market. This reveals the duality between the two fundamental concepts in financial mathematics theory.

Following Hafner [51] we applied the idea of a replicating strategy to derive a pricing formula for a variance swap. This financial contract is the building stone for newly introduced volatility indices such as VDAX-NEW, see Carr and Wu [20]. We calculated other volatility indices for different fixed time to expiry to derive a term structure of the volatility of VDAX-NEW. We saw that this term structure is almost always decreasing. This has the financial interpretation that prices from short-term options are more volatile than prices from long-term options.

One way to handle obvious short comings from simple models, such as the Black-Scholes model, is to consider a stochastic volatility process. Following Fouque et al. [46] we summarized several models considered in the literature, and presented a stochastic implied volatility model from Hafner [51] in more detail. Unfortunately the market model is not complete nor arbitrage free. However, by construction it is free from all static arbitrage opportunities and at least locally (ignoring integrability problems) arbitrage free. In this context we demonstrated Schönbuchers [79] results that the volatility of volatility measures the convexity and the correlation between the volatility and the stock price measures the minimum of the implied volatility surface. Using the theorem from Breeden and Litzenberger [16] this translates into the kurtosis and skewness of the risk neutral distribution, respectively. From a modelling point of view this is of some importance, because it allows us to measure the (unobservable) volatility of volatility or correlation through the (observable) implied volatility surface.

In the last chapter we gave a brief overview of financial models with transaction costs. In a Black-Scholes framework with proportional transaction costs and discrete revision intervals, Avellaneda and Parás [5] proved that a PDE can be derived, very similar to the well known Black-Scholes PDE. In the case of a call option this PDE simplifies to the one derived by Leland [63], which is exactly the Black-Scholes PDE with a modified volatility parameter. To get a first impression on bid/ask spreads we calculated these with the formula derived by Leland. We proved a put-call parity in models with transaction costs. Already in 1974 Gould and Galai [49] derived a similar result. Our result immediately follows form a more general result, used by Perrakis and Lefoll [71].

CHAPTER 3 Cross-Sectional Distributions of Market Maker Spreads

In 2005 Eurex introduced a new market making model, details can be found in the Eurex Market Making brochure 2005 [38] and 2006 [39].

In this chapter we will investigate cross-sectional distributions of market maker bid/ask spreads in stock index options markets. The need to understand possible influence factors is important for many reasons. On the one hand traders will be influenced by the spread either to buy the option or not, therefore the spread has an impact on the liquidity. On the other hand the goal of an exchange is to create liquidity and transparency, the understanding may therefore offer guidance in market making design, for example setting a fair maximum allowed spread for traders and market makers alike.[1] The fairness of a quote may also be interesting from a regulators point of view, i.e. are market makers extracting horrendous fees[2] for their service or are they fair considering market making costs and risks.[3]

Since 1968 starting with the work of Demsetz [28] the bid/ask spread has been of interest to researchers. Especially in the 70's a lot of research has been done on spreads for stock markets[4] followed by forex markets in the middle of the 70's (e.g. Fieleke [44] and Allan [3]). Index options were rarely investigated, e.g. by Lightfood et al. [64] in 1986 or George and Longstaff [65] in 1993. In contrast to research done in the past, we are able to differ between quotes from different market makers, we therefore can compare their quoting behaviour and hence their risk preference among each other. We do not consider best bid/ask spread nor only spreads from traded quotes, but all available quotes.

We will first introduce the markets for which we will have a closer look. After that we will present the market making model at Eurex. On the following pages we will analyse costs and risks faced by market makers to find possible explanatory variables for the distribution of the size of market makers bid/ask spreads. The argumentation was influenced by George and Longstaff [65] and Bollen et al. [13].
In chapter 3.4 we will give a brief overview of the available data, which will be used in the following chapter. We then take a closer look at the cross-sectional distribution of market makers bid/ask spreads and try to explain them with the market variables discovered before. We finish this chapter with a linear regression model and a short summary.

1. As a result of this thesis, maximum allowed spread at Eurex is in the progress of change.
2. Where we understand the fee as the difference between the bid and the ask price, i.e. the spread.
3. For a summary of why "understanding the determinants of the market maker's bid/ask spread is important" see Bollen et al. [13].
4. Bollen et al. [13] gives a good summary of the research done on stock markets.

CHAPTER 3.1 Market Making at EUREX (2005)

3.1.1 Markets

DAX (Deutscher Aktienindex) measures “the performance of the Prime Standard's 30 largest German companies in terms of order book volume and market capitalization” [30]. It was launched on 1 July 1988 at Frankfurt Stock Exchange, the base date is set to 30 December 1987 with 1000 points. It is a free float weighted, performance index, i.e. dividends are reinvested. Currently it is calculated every second from 9:00 a.m. till 5:30 p.m. CET on every trading day in Frankfurt. It is calculated from prices generated in the electronic trading system Xetra using the Laspeyres formula.

Options on this index were first launched in August 1991 on the Deutsche Terminbörse (DTB) (predecessor of Eurex). They have a contract value of EUR 5 per index point and the minimum price change is 0.1 index points or EUR 0.50, respectively. The minimum quote size is 50 contracts.
Currently market makers have to quote at least 80 different DAX options, consisting of put and call options for eight different maturities covering up to 24 months[1] and of at least 5 different strike prices around the current index level. All options are European style options, i.e. they may only be exercised on the final settlement day, where the final settlement day is defined to be the third Friday of each maturity month, if this is a trading day, otherwise the closest exchange trading day before.

Currently 75 different market makers have a licence to quote DAX options, and about 20 of them fulfilled their coverage obligations in March 2006.[2] This ensures a very competitive market. There are in average 221,520 options traded on the DAX per day, which makes a 12% share of the total average traded equity and equity indices options per day of 1,880,831 (see table 3.1.1).

ESX (Dow Jones EURO STOXX 50), was introduced at the beginning of 1998. It “tracks the performance of the 50 most important and most actively traded companies in the euro zone” [29]. The base date is 31 December 1991 with 1000 points. It is calculated every 15 seconds, during trading hours.

Options on the ESX are provided with the same settings as options on the DAX, except that the contract value is set to EUR 10.

1. Which are the three nearest successive months, the three following months of the cycle March, June, September, December, followed by the next two months of the June and December cycle.
2. See chapter 3.1.2 for a definition of “coverage”.

SMI (Swiss Market Index) “was introduced on 30 June 1988 at a baseline value of 1500 points” [80]. It consists of up to 30 of the largest and most liquid SPI large- and mid-cap stocks. It is calculated in real-time, i.e. after every transaction, and published every three seconds.

Options on the SMI are provided with the same settings as options on the DAX, except that the contract value is set to CHF 10.

Table 3.1.1 Summary of Market Details[a]

Market	Average Contracts Per Day[b]	Put/Call Ratio	Share From Total Traded Options[c]
DAX	221,520	1.2	12%
ESX	488,041	1.68	26%
SMI	13,874	1.45	1%

a. for details see [42].
b. daily average from 1 January 2006 till 30 March 2006.
c. total traded stock index and single stock options at Eurex.

3.1.2 Market Makers

Market makers are providing liquidity and transparency by entering bid/ask prices for certain products, e.g. options on the DAX. In fact a possible definition for market maker is “a trader who is willing to quote both bid and offer prices for an asset” [55]. These entered quotes are binding for the market makers and all market participants may execute orders for these prices.

Eurex offers three different kinds of market making models, regular (RMM), permanent (PMM) and advanced (AMM) market making. For our purposes only PMM and AMM market makers are important. The market making model changed in 2006, so that now RMM “applies only to less liquid options on equities, equity indexes...” [39]. Therefore RMM licences are not available for options on DAX, SMI, or Euro Stoxx 50 index since 2006.
PMM and AMM have to quote options continuously for all maturity days up to 24 months, i.e. eight different maturity days, and five out of seven different strike prices around the underlying price for an average of 85% between 8:50 am and 5:30 pm. Only valid quotes are considered in calculating the coverage, where a quote is valid if minimum quote size (see table 3.1.2) and maximum spreads obligations are fulfilled (see chapter 3.1).
Whereas PMM only have to quote some predefined products, AMM have to quote all products of a predefined package, which could be for example “all German equity options listed at Eurex and the DAX® Options” [39].

If market makers fulfil their coverage obligations on a monthly average they are rewarded with a partial transaction fee refund for providing liquidity.

We compare four different maker makers, called MM1, MM2, MM3 and MM4. We selected the three best market makers from all 14 market makers with a PMM or AMM licence in quoting DAX, SMI and ESX options. This implies that in average quotes from MM1 to MM3 are most likely to be executed.[1] The fourth market maker (MM4) was taken into consideration, because the quoting behaviour was known a priori, and could therefore be used as a benchmark and for verification purposes. Market makers sometimes add more than five million quotes for one underlying per day, as happened on 16 May 2006 for DAX options by MM2, which would be roughly one quote per second for every series they would have to quote.

All four market maker have an AMM licence for quoting DAX options and except for MM2 for ESX options. Except for MM4 all have a PMM licence for quoting SMI options in April 2006.

Table 3.1.2 Summary of Contract Details[a]

Market	# Market Makers	Contract Value	Minimum Tick [Index Points]	Transaction Costs PMM (AMM)	Execution Costs PMM (AMM)	Quote Size
DAX	75	EUR 5	0.1 (EUR 0.5)	EUR 0.20 (0.10)	EUR 0.20 (0.10)	50
ESX	60	EUR 10	0.1 (EUR 1.0)	EUR 0.15 (0.08)	EUR 0.15 (0.08)	50
SMI	43	CHF 10	0.1 (CHF 1.0)	CHF 0.30 (0.22)	CHF 0.30 (0.22)	50

a. Costs listed are after market makers refund, see [41] for Eurex fees.

3.1.3 Maximum Spread Size And Spread Quality

Maximum Spreads

One obligation for a valid spread is that it has to be below a specific maximum spread, which depends on the bid price, the underlying market and the current market state.

Maximum allowed spread size for DAX and ESX options are defined as:

Definition 3.1.3 (Maxspread) Maximum Allowed Spread for DAX, ESX Options
Maximum allowed spread for DAX, ESX Options measured in index points.

$$\text{maxspread} := \begin{cases} 1.4 & \text{bid} \le 14 \\ \frac{1}{10}\text{bid} & \\ 13.4 & \text{bid} \ge 134 \end{cases}$$

and for SMI options:

1. Best market makers according to average spread quality from quotes on DAX, SMI and ESX options for the first three months in 2006.

Definition 3.1.4 Maximum Allowed Spread for SMI Options

Maximum allowed spread for SMI Options measured in index points. Floor increased to 35 and cap to 350 in market making model 2006.

$$\text{maxspread} := \begin{cases} 2.7 & \text{bid} \le 26.60 \\ \frac{1}{10}\text{bid} & \\ 26.7 & \text{bid} \ge 266.60 \end{cases}$$

As a consequence maximum allowed spread therefore moves with the bid price until it reaches a cap or floor. During fast market periods the maximum spread size is doubled.[1]

If we look at regularly traded options on the DAX index we see that the caps and floors are very restrictive, i.e. the bid price easily rises above $\overline{134}$ index points or below $\underline{14}$ index points (see table 3.1.5).

Table 3.1.5 Example DAX Option Premiums

Strike	Days To Expiry	Call (Put) Option Premium (with spot = 6000, volatility = 15%)
5800	10	$\overline{213.07}$ ($\underline{5.13}$) [Index Points]
6000	40	$\overline{135.64}$ (102.85) [Index Points]
6200	10	$\underline{7.41}$ ($\overline{198.92}$) [Index Points]

The introduction of a maximum spread size complicates the analysis of market maker spreads in several ways:

First, by definition 3.1.3 it induces an artificial correlation between bid price and market maker spreads, which can be seen in figure 3.5.3 and figure 3.5.4.

Second, it forces the market maker to quote in a specific range. It is therefore rather difficult to predict anything about his quoting behaviour for very expensive options.

Third, we cannot easily compare spread quality as defined by definition 3.1.6 between different options, because the spread quality for very expensive options are artificially decreased and for cheap options increased.

Spread Quality

The spread quality of a quote should be a measure of how good a quote is from the traders point of view. One natural definition would be to define the spread quality as the relation between the spread and the maximum spread:

Definition 3.1.6 Eurex Spread Quality

$$E(\text{ask}, \text{bid}) := 1 - \frac{(\text{ask} - \text{bid})}{\text{maxspread}}$$

We call $E(\text{ask}, \text{bid})$ the *spread quality*, where ask is the ask price, bid the bid price of the quoted option and maxspread is the maximum allowed spread defined in definition 3.1.3

1. "Fast Market is a special trading phase activated by Eurex Market Supervision on a per-product basis in certain circumstances, such as high volatility or special events. During this phase, the validation of quote entry is subject to more relaxed parameters. Otherwise the same facilities are available as those in the Trading Period." [39]

But as noted before this definition may be misleading for expensive or very cheap options.

Instead of using a spread quality relative to the maximum allowed spread, we also use a spread quality relative to the bid price (definition 3.1.7).

Definition 3.1.7 Spread Quality

$$E(\text{ask}, \text{bid}) := 1 - \frac{10(\text{ask} - \text{bid})}{\text{bid}}$$

We call $E(\text{ask}, \text{bid})$ the *spread quality*, where ask is the ask price, bid the bid price of the quoted option and maxspread is the maximum allowed spread defined in definition 3.1.3

This on the other hand has the disadvantage that the spread quality will converge to 1 for $\text{bid} \to \infty$ (independent of the spread), because the spread is still being capped. We therefore can compare cheap options with other options, but the spread quality for very expensive options is artificially increased.

As both definitions are flawed, we put our main effort in an analysis of the spreads themselves, rather than any relative spreads.

3.1.4 Summary

We saw that the caps and floors of the maximum spread are very restrictive and that a maximum spread complicates matters in several ways. The maximum spread makes it difficult to give a satisfying definition of spread quality and it induces an artificial correlation between spread and bid price.

CHAPTER 3.2 Spread Management Strategies

Market makers mainly have to pay all arising costs from the profit they make with trading options in and out, i.e. buying and selling the same option.
We will discuss three different cost categories following Bollen et al. [13]: inventory-holding, adverse selection and order processing costs. We will divide these cost categories into risky costs and deterministic costs.
First we will look at possible risks market makers have to face and then give a summary of expected, deterministic costs.

3.2.1 *Market Making Risks*

Inventory-Holding Risk. Let us write $ask(t), bid(t)$ for the ask and bid price given by a market maker at a certain time t and $theo(t)$ for the theoretical/fair price of an option at time t.
Assuming symmetric pricing[1] and constant spreads, we can write:

$$ask(t) = theo(t) + \frac{1}{2}spread$$

and

$$bid(t) = theo(t) - \frac{1}{2}spread$$

If a market maker first sells an option, i.e. he becomes the writer of the option, at time $t = 0$ for $ask(0)$ and then closes the option by buying the same option, i.e. he becomes the holder, at time $t = T$ for $bid(T)$, we can calculate his profit as

Formula 3.2.1 Sell/Buy Profit

$$profit = spread + theo(0) - theo(T)$$

Analogously we can calculate the profit for a market maker if the trade to buy the option occurs first:

Formula 3.2.2 Buy/Sell Profit

$$profit = spread + theo(T) - theo(0)$$

We therefore can see the importance of the holding time T, if the holding time is close to zero market makers do not have to fear inventory holding risk and a profit from a round trip is guaranteed. Otherwise the option can fall or rise in value and therefore result in a loss for the holder or writer of

1. Pricing is usually asymmetric, see Nordén [68].

the option, respectively. Especially for illiquid options this risk will probably play an important role in how big market makers set their spreads.
In the next chapter we will take a closer look at inventory holding risk and how to measure it.

Adverse Selection Risk. Because information about the expected price movements of the option is not equal between all market participants, the risk of dealing with an informed trader, i.e. a trader with more information than the market maker, is substantial.
But because all indices considered consist of more than 30 different stocks, insider information on one stock will not have such a big impact on the index as on the stock itself. Also because the trading volume of all indices is huge[1], the activity of uninformed traders is very high.[2] Therefore the risk is probably marginal and, as a consequence we will ignore it.

3.2.2 Market Making Costs

Order Processing Costs are the costs for providing the market making services. Among these are fixed costs (salaries, rent, and others) and order processing costs (transaction fees, clearing fees). They should therefore fall with the trading volume, because the fixed costs can be divided among more trades. Market makers usually quote in several different markets. Therefore this relation will probably be not visible in one market. Hence we will ignore trading volume as an explanatory variable.

Inventory-Holding Costs are the costs arising from the funds tied up in the market makers' inventory. This could just be the value of the option being bought or also a margin account required by their clearing firm.[3] For example every Eurex member with a clearing licence has to pay a contribution to Eurex Clearing AG's clearing fund, this however is fixed and depends on the clearing licence.[4]
The inventory holding cost should therefore be correlated with the options bid price.

3.2.3 Practical Depending Influence Factors (Exchange Rules)

George and Longstaff [65] also consider a dummy variable which is 1 if the bid price is greater than or equal to three dollars. This is because the S&P 100 options at CBOE have a different minimum tick size, depending on the bid price.
At Eurex the minimum tick size is the same for all options on individual markets, which allows us to ignore the minimum tick size. Instead we introduce two dummy variables to measure the influence of the floor and cap to maximum spread size (see definition 3.1.3).

1. See table 3.1.1.
2. Easley et. al. [34] use the volume of trading as a measure for Adverse selection risk.
3. George and Longstaff [65] pointed out that clearing firms sometimes require a margin account.
4. For details, see Eurex brochure [37].

3.2.4 Theoretical Depending Influence Factors (Put-Call Parity)

George and Longstaff [65] point out, that the put-call parity lemma 2.1.36 $C_t - P_t = S_t - Ke^{-r(T-t)}$ can be used to derive a theoretical spread on the underlying and that this can be compared to observed spreads on the underlying.
They argue that a spread from a call $\Delta C_t(K, T)$, from a put $\Delta P_t(K, T)$ and from a bond $\Delta B_t(K, T)$ summed up has to be equal to the spread on the underlying ΔS_t:

$$\Delta C_t(K, T) + \Delta P_t(K, T) + \Delta B_t(K, T) = \Delta S_t$$

Considering the following argumentation, the result of George and Longstaff is not an implication of the put-call parity:

Adding (EQ 6, p. 73) and (EQ 7, p. 73) only gives us:

$$\Delta C + \Delta P \geq -\Delta S - K(e^{-\beta(T-t)} - e^{-\alpha(T-t)}) \qquad \text{(EQ 1)}$$

where the right side of (EQ 1) is negative. Because the left side of (EQ 1) has to be positive, this does not imply any relations between the options spreads and the underlying stock spread.

We therefore cannot prove that a connection between underlying spread and option spreads must exist.

3.2.5 Summary

So far we have identified four possible influence factors for market makers spread size, which are the holding risk, the bid price and two dummy variables, which test whether the maximum spread is capped or floored.

The question remains how we will model market makers' inventory holding risk, i.e. we need a measure for T, i.e. the time at which an offsetting trade arrives, and a measure for the risk involved holding an open position till time T.

We will investigate both in the next chapter.

CHAPTER 3.3 Inventory Holding Risk

3.3.1 Fixed Time to Offsetting Trade

Let us first assume that the time to offsetting trade T is fixed. In formula 3.2.1 and formula 3.2.2 we noticed that the only risky factor in the market makers' profit from a round trip is the value given by $\mp[\text{theo}(T) - \text{theo}(0)]$. To nullify just the risk, i.e. to make the profit greater than or equal to 0, we would need a contract with a payoff like $\max\{0, \pm[\text{theo}(T) - \text{theo}(0)]\}$, which is the payoff from an at the money compound option with time to expiry T (see definition 2.2.8).
We saw that a compound option could be used as a measure for the holding risk. This approach is similar to the one used by Bollen et al. [13] to measure the holding premium on Nasdaq stocks by an at the money option.

Other risk measures include the variance of option price changes: $\text{VAR}(\text{theo}(T) - \text{theo}(0))$, which simplifies to $\nu\delta^2$ in the Black-Scholes framework, where ν is the volatility of the underlying asset and δ is the Black-Scholes delta of the option (as noted in George and Longstaff [65]). This risk measure is used by George and Longstaff [65], who further argue that since the "same volatility applies to all the options [...] cross sectional differences in squared deltas capture differences in the risk of holding uncovered inventory positions".
However, this procedure is questionable for two reasons. First, as pointed out by Brachinger [15], "risk is the chance of something bad happening" and therefore the variance should be replaced by a more appropriate measure like for example the semi-variance $\text{VAR}(X|X < 0)$. Second, although in the Black-Scholes framework the volatility is the same for all options, in "reality" this assumption does not hold, as already mentioned in chapter 2.3.2.[1]

Summary

A very good upper border risk measure would be an ATM compound option. However, in the Black-Scholes framework the price structure of a compound option is very similar to delta[2], which can also be thought of as the variance of an option price process. For this reason we will consider delta squared as our risk measure and follow George and Longstaff [65].
Even though this only grabs one risky influence factor, namely the underlying stock price process, it seems to be the most important one in consideration done by market makers.

1. An example implied volatility smile can be seen at figure 3.5.10.
2. See chapter 2.2.6.

3.3.2 Variable Time to Offsetting Trade

Under real world conditions the time till an offsetting trade arrives T will be a random variable. T will depend on the liquidity of the option, which depends on the option type, the volume, the exercise price and the time to expiry.

If we look at the Xetra Liquidity Measure (XLM)[1] the liquidity decreases with time to expiry, volume and with moneyness. Also, because put options are normally traded more than call options (see table 3.1.1) we should include the option type in our consideration.
The influence of the moneyness to the liquidity can be ignored because we already identified squared delta as an appropriate risk measure. Also we will ignore the quote size, because no influence could be measured (see figure A.7). This can be explained by the fact that all quote sizes were relatively small. The maximum quotes size was found to be 1600 contracts.

Summary

We consider the option type and the time to expiry as a simplified measure for liquidity.

1. For a definition see [48].

CHAPTER 3.4 Review of Available Data

Data Set 1. At the beginning of this thesis only one type of data was available: The averaged spread quality defined in definition 3.1.3 per day, market and marker maker since 22 March 2005, i.e. since the new market model was introduced in 2005.

Table 3.4.1 Example Data (Dataset 1)

Date	Market Maker	Market	Spread Quality
6 Mar 2006	MM1	DAX	0.57

Because the 1st data set could not grab the different features for different options, upon my request, the data has been split up to calculate the spread quality for every series independently:

Data Set 2. Contains the average spread quality per series, day, market and market maker since 3 April 2006 to 31 July 2006. For a summary of the data see table A.5.

Table 3.4.2 Example Data (Dataset 2)

Date	Market Maker	Market	Type	Strike	Expiry Date	Spread Quality
10 Apr. 2006	MM1	DAX	Call	6000	200604	0.48

In the 2nd data set an artificial correlation was induced by the definition of the maximum spread, which could not be filtered out because no bid prices were available. Also caps and floors make it more difficult to work with spread quality rather than the spread itself.

We therefore mainly look at quoted ask and bid prices. To increase the data set we will consider intraday data. Because of the vast amount of market maker quotes per market on a daily basis, the data had to be aggregated. An aggregation interval of 10 minutes seemed to be reasonable, reducing the amount of data significantly and still being able to capture probably all intraday effects.

Data Set 3. We will look at market makers' quoted call and put options for DAX, SMI and ESX indices on 4th, 5th, 7th, 10th and 18th through 21st of April 2006. For a summary of the data see table A.4.
All quotes are time averaged over 10-minute intervals. An example of the data can be seen in table 3.4.3

Table 3.4.3 Example Data (Dataset 3)

Date	Time	Market Maker	Market	Type	Strike	Expiry Date	Ask	Bid	Ask Size	Bid Size
7 Apr. 2006	16:00	MM1	DAX	Call	6000	200604	61	60	50	50

We also have the underlying intraday prices from which we calculate a log-simple moneyness defined by:

Definition 3.4.4 Log-Simple Moneyness

$$M := \log\left(\frac{\text{Strike}}{\text{Spot}}\right)$$

We therefore call an option at the money (ATM) if its strike is equal to the spot. This is technically not correct, an option should be called ATM if its strike is equal to the forward as in definition 2.3.11. It is used in Exchanges, because it is straight forward to implement. We follow this definition for consistency.

We divide the data into maturity categories, as defined in table 3.4.5 and moneyness categories defined in table 3.4.6.

Table 3.4.5 Maturity Categories

Maturity Category	Days to Expiry
1	< 30
2	< 90
3	< 180
4	< 270
5	≥ 270

Table 3.4.6 Strike Categories

Strike Category	$M := \log\left(\frac{\text{Strike}}{\text{Spot}}\right)$	Call	Put
–1	< -0.1	In The Money (ITM)	Out The Money (OTM)
0	$-0.1 \leq M \leq 0.1$	At The Money (ATM)	At The Money (ATM)
1	> 0.1	Out The Money (OTM)	In The Money (ITM)

Additionally we have all intraday volatility subindices from DAX and ESX options as defined on page 45 at our disposal. As shown on page 49 it is reasonable to evaluate options with this specific volatility. We therefore calculate the Black-Scholes greeks for each option using the specific subindex as the volatility input parameter. The advantage to options implied volatility is that we have an objective theoretical price and greeks. We therefore can compare both easier among market makers. The disadvantage is that we assume a flat smile, and hence consider slightly wrong delta values mainly for short term, ITM or OTM options.

Note that we consider actual market maker quotes and not quotes from traded options nor best bid/ask quotes only. Because no Eurex regulation[1] exists that enforces market makers "to compete with other market makers to improve markets in all series" (CBOE rule 8.7)[2] the market makers quote can and will differ very much from actual traded quotes.

Only valid quotes will be considered in our calculations, i.e. quotes with a spread less than maximum allowed spread, as defined in chapter 3.1, and a quote size higher than minimum quote size (see table 3.1.2).

Because no significant difference could be found between different markets all observations are made from DAX options.[3] All results from data set 3 will then be re-evaluated using data from data set 2.[4]

1. For more details about EUREX regulations see [40].
2. For more details about CBOE regulations see [22].
3. See appendix A.8 for observations on ESX.
4. See appendix A.7.

CHAPTER 3.5 Market Maker Spreads And Spread Quality

Market Maker spreads are far from constant over option series, they reach from 1.0 index points up to the maximum allowed spread of 13.4 index points, which in currency value is equivalent to EUR 5 or EUR 77 for DAX options, respectively. Also the relative spread is far from constant, reaching from almost 90% to strongly negative values (definition 3.1.7), respectively a spread quality of zero (definition 3.1.6).
The total average spread from the 3rd data set on DAX options is about 5.9 index points (EUR 29.5) for calls and 5.4 index points (EUR 27) for puts (ignoring quotes from MM4).

3.5.1 Market Maker Spreads

After investigating the relevant factors, we will first take a look at the spread size by bid prize of all quotes on DAX options on two specific days (April, 4th and 7th):

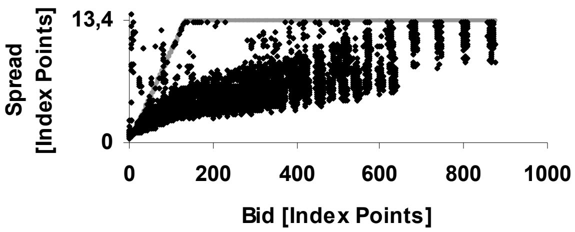


Figure 3.5.1 MM1 Spread Versus Bid Price.

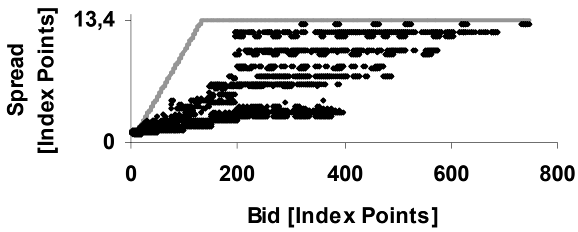


Figure 3.5.2 MM2 Spread Versus Bid Price.

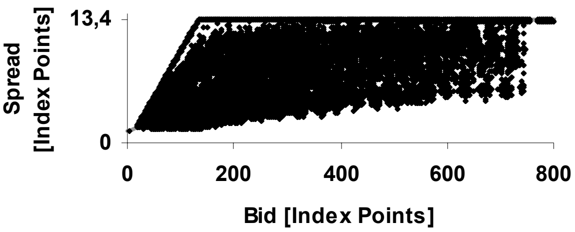


Figure 3.5.3 MM3 Spread Versus Bid Price.

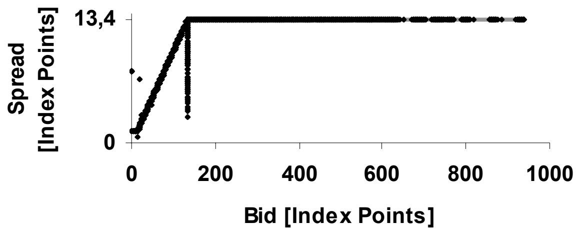


Figure 3.5.4 MM4 Spread Versus Bid Price.

Several similarities and differences between different market makers are notable. The main similarity is that all four market makers increase their spread size with increasing bid price, but for MM1 and MM2 this is independent of maximum spread size (grey line), whereas the maximum spread size of

10% of bid induces an artificial correlation partially for MM3 and totally for MM4. The latter just follows the maximum spread size, with the only exceptions for options with a bid price of 134 index points, i.e. the upper border of the maximum spread size.[1] MM3 quotes similar for all different bid prices, also note that MM3 never quotes options with a bid price less than 13.4, i.e. the floor on maximum spread does not influence MM3 spreads. MM2 almost quotes in rows, i.e. one spread size for a range of bid prices. MM1 could be said to quote in columns, i.e. for each bid price there is a range of different spread sizes.

The question to be asked is what kind of options creates these columns and rows.

But before we investigate this any further, we first look at the intraday changes in spread size for an example option:

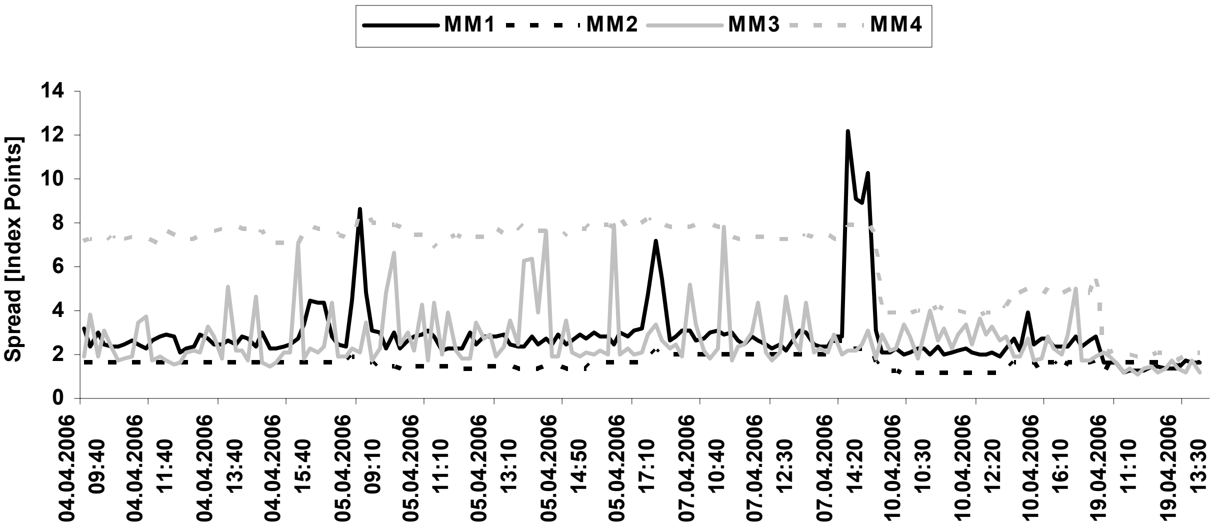


Figure 3.5.5 Intraday Spread Changes

Spread changes for a DAX Call-6000 option with expiry 200604 (dataset 3).

As can be seen in figure 3.5.5 MM3 (grey, solid line) changes its spread size very frequently reaching from as low as 2 index points up to the maximum spread size of 10% of the bid price. This can explain the 2-dimensional structure of the previous plot (figure 3.5.3). As mentioned before MM4 (grey, dotted line) just follows the maximum spread size. MM1 (black, solid line) and especially MM2 (black, dotted line) seem to quote with an almost constant spread size at a very low level. MM1 only increases the spread size during market opening and closing times. No possible influence factor identified on page 84 is as volatile as the spread size from MM3, therefore we can conclude that these changes cannot be explained with any one of them.[2] Because of the almost constant spread size from MM1 and MM2 for each option, the dependency between the possible influence factors and the spread size can best be verified for these market makers.

1. This exception is due to a calculation error. The market maker has been informed about this issue and he confirmed that this was not intended.
2. The frequent jumps up to maximum spread size, and the immediately following drop to the previous quoting level, was explained to me on a personal visit to market maker 3. It is a model specific reason and not of analytical interest. The precise information is restricted and not for publication.

Let us now look at the cross-sectional distribution of spreads. figure 3.5.6 divides all options by option type (call, put), maturity category (table 3.4.5) and moneyness category (table 3.4.6).

The average spread size over all available data is calculated for each category. Because no significant difference between MM1, MM2 and MM3 could be found the average is taken over all spreads from these three market makers.

As we can see for short term options the spread size increases for in the money options. This is flattening out for longer term options. Also a strong symmetry between call and put short term options can be seen. In general the spread size also increases with time to maturity, except for in the money short term options.[1]

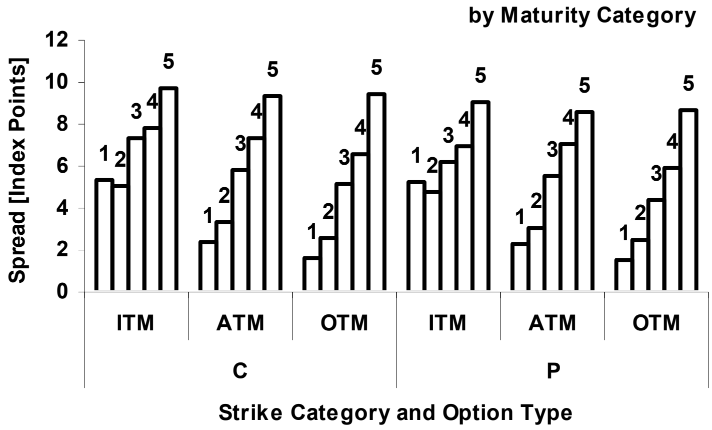


Figure 3.5.6 Spread Distribution

Average Spread from all quotes on DAX options from MM1, MM2 and MM3 (dataset 3).

Short term ITM. Let us take a closer look at this exception by plotting the spread size for each available day (figure 3.5.7).

As can be seen here, the spread is increasing for in the money options of MM1 and MM3 while approaching the expiry day (21 Apr. 2006). For MM2 the difference was rather small, and only a more comprehensive data set could give us reliable results. Because spreads of MM4 follow the bid price, the spreads are decreasing while approaching the expiry day.

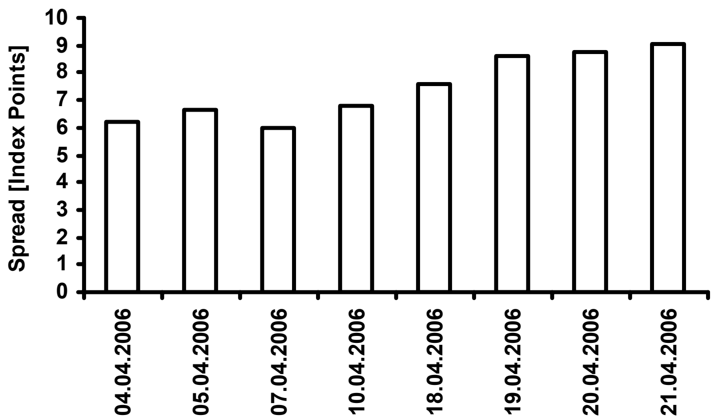


Figure 3.5.7 Spreads Before Expiry

Average spread from ITM DAX options expiring on 21.04.2006 from MM1 and MM3 (Dataset 3).

As seen in chapter 2.2.5 the delta of a call option is

$$\delta_c(K) = \mathbf{N}(d_K)$$

and for a put

$$\delta_p(K) = \mathbf{N}(d_K) - 1$$

1. This exception for short term ITM options was also observed with data from data set 2, see figure A.8.

with:

$$d(\tau) = d_K = \frac{\log\left(\frac{S}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}} \qquad \text{(EQ 1)}$$

If now $\log(S/K) > 0$ we can see that $d(\tau)$ is decreasing for low τ [1], i.e. it is increasing for decreasing time to expiry τ, and therefore – because the normal cumulative distribution function $\mathbf{N}(x)$ is increasing in x – the squared delta of in the money call options is increasing with decreasing time to expiry.
If now $\log(S/K) < 0$ we can see that $d(\tau)$ is increasing (for all τ)[1]. Consequently the delta of in the money put options is decreasing with decreasing time to expiry. Because the delta is negative for a put option, the squared delta is therefore also increasing with decreasing time to expiry.
Hence we can explain figure 3.5.7 with the spread size following squared delta.

Another explanation could be that market makers charge a exercise fee for in the money options close to expiry, to compensate Eurex additional exercise fee which is at least EUR 0.10 per contract.[2]
This would also explain why this feature was not observed by George and Longstaff [65], because they considered American options, which can be exercised at any time.[3] On the other hand an execution fee of less than EUR 0.20, which is less than the minimum tick size, should have only a marginal influence.

Symmetry Call/Put. Let us now try to explain the symmetry between call and put options spread.
If we only consider short term options, we can ignore the term

$$\frac{1}{\sigma}\left(r + \frac{1}{2}\sigma^2\right)\sqrt{\tau}$$

in (EQ 1) because it will be very close to 0, e.g. 0.05 for an interest rate of 5%, a volatility of 15% and time to expiry of 5 days. Now calculating the delta of a call option with $\log(S/K) = -a$ yields:

$$\delta_c(e^a S) = \mathbf{N}(d_{e^a S}) = 1 - \mathbf{N}(d_{e^{-a} S}) = -\delta_p(e^{-a} S)$$

where we used $\mathbf{N}(-x) = 1 - \mathbf{N}(x)$ in the second equality. Therefore the squared delta of short term call option with strike cS is equal to the squared delta of short term put option with strike S/c. Thus we can explain the symmetry between short term call and put options in figure 3.5.6 with the spread size following squared delta.

1. $d(\tau)$ is increasing if and only if $\tau \geq 2 \cdot \log\left(\frac{S}{K}\right)\left(r + \frac{1}{2}\sigma^2\right)^{-1}$.
2. See table 3.1.2 (p. 79).
3. Another explanation could be that because they consider quotes from all market makers in the open outcry process, at least today, CBOE rule 8.7 (b) (iv) (A) [22] states a maximum spread depending on the bid price. If we would also consider market makers who follow maximum spread this feature would not be visible.

Because weekly options are expiring on the 1st, 2nd, 4th and, if available, 5th Friday of a calendar month these options are short term options. As a consequence increasing spread close to expiry and a symmetry between a call and a put with inverse strike should be very strong for weekly options.

For long term options another interesting feature between call and put options can be found. George and Longstaff [65] found a correlation between call and put option spreads with the same strike and maturity. We found the same correlation in all three markets, see table 3.5.8. The correlation is very strong for medium and long term options from MM1 and especially for ATM options from MM3.[1]

Table 3.5.8 Average Correlation (Medium Term)

Average correlation (formula A.2) between call and put spread of the same strike for all options (in brackets only ATM options) in expiry series 2, 3, 4 and 5 from series with at least 200 quotes (dataset 3).

[%]	ODAX	OSMI	OESX
MM1	72 (70)	45 (44)	58 (55)
MM2	9 (4)	51 (35)	34 (31)
MM3	36 (62)	17 (21)	45 (59)
MM4	-57 (-52)	-	-59 (-63)

A strong negative correlation could be found for all short and medium term options from MM4, which can be explained by the fact, that MM4 is following maximum allowed spread and therefore the bid price.[2] Put and call parity lemma 2.1.36 implies that rising call prices result in decreasing put prices. The correlation is strong for MM2 for long term options as can be seen in table 3.5.9. Put and call parity also shows us that market makers, who are long a call and short a put, only have to fear underlying movements, i.e. their portfolio has a delta of 1 and gamma and vega are 0.[3] Therefore it makes sense that some market makers seem to see calls and puts with the same strike and time to expiry as substitutes.

Table 3.5.9 Average Correlation (Long Term)

Average correlation between call and put spread of same strike and expiry series 6, 7 and 8 from series with at least 200 quotes (dataset 3).

[%]	ODAX	OSMI	OESX
MM1	65	All series have less than 200 quotes	69
MM2	99		80
MM3	39		34
MM4	div by 0[a]		div by 0[a]

a. See footnote 2.

1. The correlation for ATM call and put options could be explained by a similar squared delta, while the correlation for other options from MM3 is arbitrary, as was confirmed by MM3.
2. For long term options no correlation could be calculated, because the spread is constant to the capped maximum spread. Therefore the variance of the spread is 0 this would result in a division by 0, see formula A.2.
3. See table 2.2.6 for details on Black-Scholes greeks.

Even though we do not have the exact interest rate[1] and therefore the implied volatility is not exact, a remarkable figure arises if we plot the implied volatility from ask and bid prices (figure 3.5.10). As can be seen there the implied volatility from ask prices is very smooth and the same shape as a theoretical smile in a stochastic volatility model.[2]

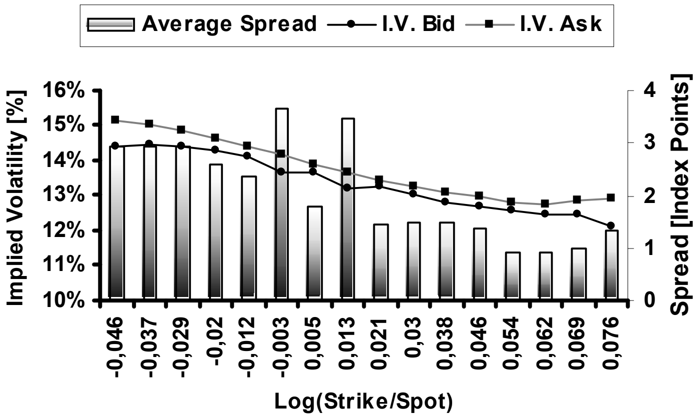


Figure 3.5.10 Spread versus Implied Volatility

Spread (columns) and Implied Volatility (lines) from DAX Call 200605 options on 10/04/2006 from MM2 (dataset 3).

It seems as if the bid price was adjusted to the ask price, which is unexpected, because the way the maximum spread is defined (as 10% of the bid price) it may be more handy to adjust the ask price to the bid price, rather than the other way around.

It almost seems that the turning point (from a convex to a concave function) of the implied volatility smile at around $\log(\text{Strike § Spot}) = -0,02$ is induced by maximum spread size. But because the spread is far away from the maximum spread size of $13,4$ for all negative x-values, this is not the case.

The fact we observed the spread being highest for ITM call and ITM put options could explain part of the smile surface. It would be revealing to analyse the impact of maximum spread or the actual spread to the implied volatility surface more deeply. Especially with lemma 2.5.6 in mind from which we can infer that the implied call volatility does not have to be equal to the implied put volatility.[3]

Summary

We saw that most characteristics of the spread surface can be explained with an absolute or squared delta. Features which cannot be explained with delta, like an increasing spread for ITM options by increasing time to expiry, can be explained with a decreasing liquidity for long term options. This is in accordance to our previous theoretical considerations where we already identified squared delta as an appropriate risk measure and the time to expiry as a simplified measure for the holding time.

3.5.2 Market Maker Spread Quality

Instead of using the absolute spread, we will now consider the relative spread, i.e. the spread quality as defined in definition 3.1.6. In figure 3.5.11 we plotted the spread quality since inception of the new market making model in 2005. Fast market occurred on 18 April 2005 and 7 July 2005.[4] The first fast market did not move any market maker to quote wider, this is why the spread quality jumped up to

1. We assumed a constant interest rate of 5%.
2. See Fouque [46], page 55.
3. See Hafner [51], pp. 50-51, and the references cited there.
4. For a definition of “fast market” see footnote 1, p. 80.

almost 90% for MM1 at April, 18th. During the second fast market period (London terror attacks) the spread widened for MM1 and MM3, MM2 did not react.

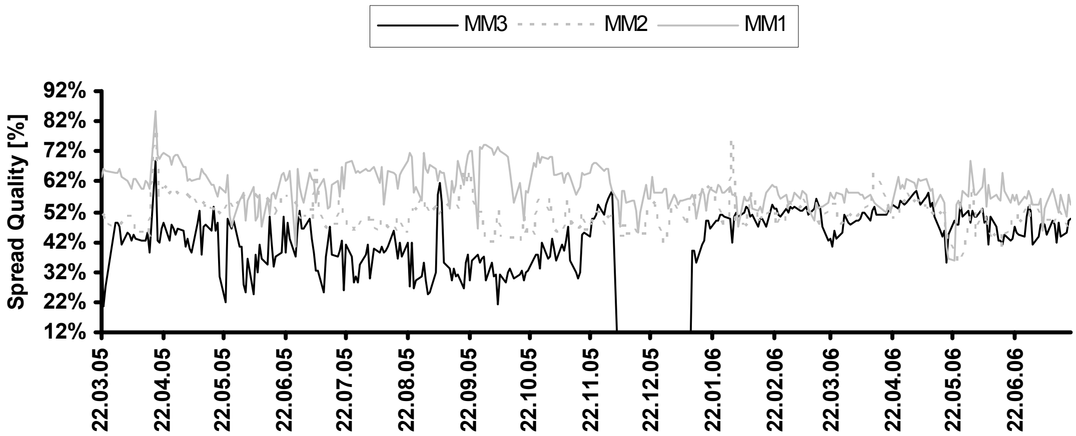


Figure 3.5.11 Daily Spread Quality

Spread Quality (definition 3.1.6) from data set 1.

The new market making model started in 2005 and it is striking that after one year the spread quality from all three market makers seemed to converge to a mean between 50 and 60 per cent.
To analyse the spread quality per series we have to go back to use data set 2 or 3.

Let us start with the intraday data set 3:
If we plot the spread quality as defined in definition 3.1.6, we get the impression that the spread quality gets worse with increasing time to expiry. This may be misleading, because the amount of good quotes, as defined later on, is still very high for options up to an expiry of one year, see figure 3.5.13.
The ratio of good versus bad quotes is increasing with increasing bid price, up to an expiry of about one year. After that the amount of bad quotes is increasing again. Also note that even though the spread for OTM options was lowest, the spread quality is lowest, too.

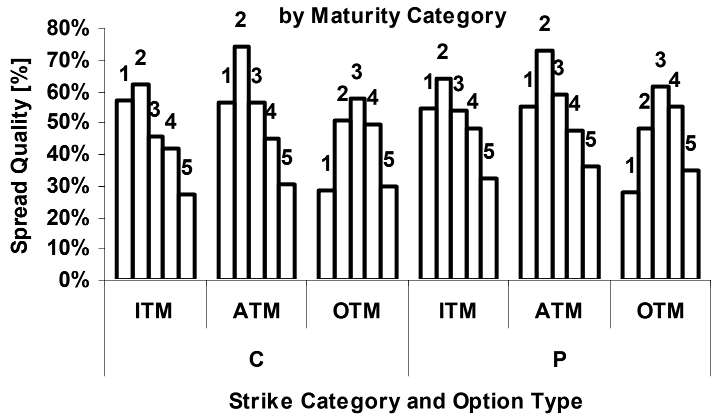


Figure 3.5.12 Spread Quality Distribution

Average spread quality (definition 3.1.6) from all quotes on DAX options from MM1, MM2 and MM3 (dataset 3).

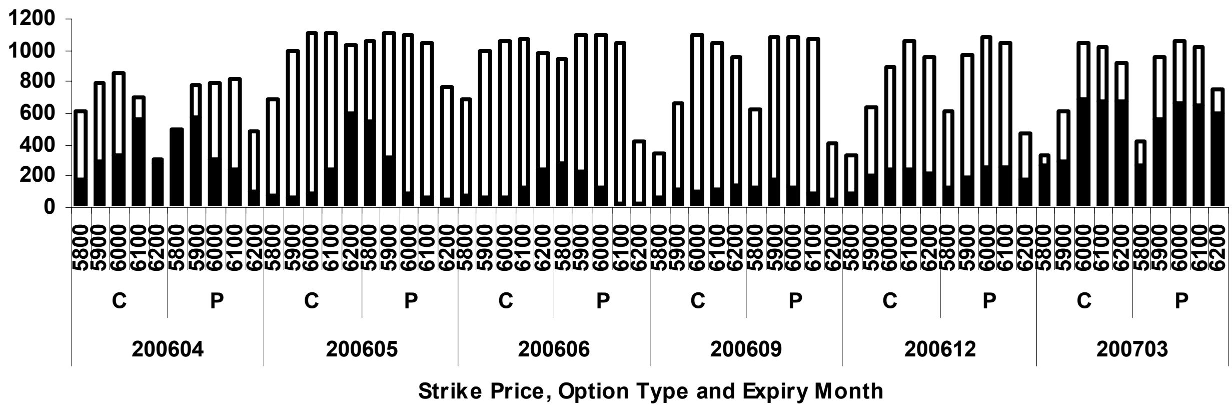


Figure 3.5.13 Good/Bad Quotes

Number of 10-minute intervals with good quotes (white), which are quotes with a spread of less than half maximum spread, and bad quotes (black) from MM1, MM2 and MM3 on DAX options (dataset 3).

Data set 2 does not reveal any new informations, we will just plot it for the sake of completeness:

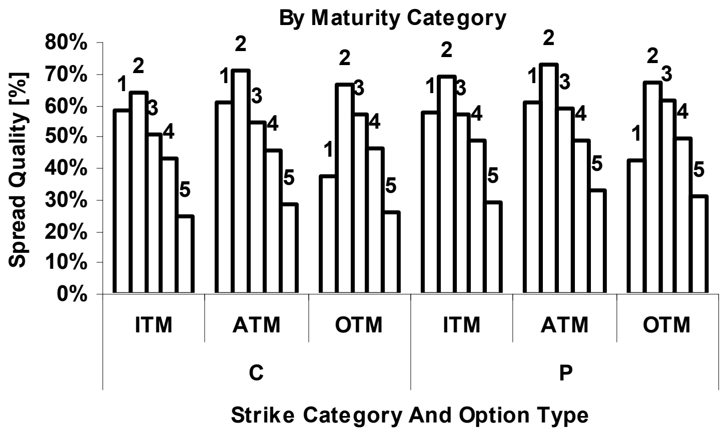


Figure 3.5.14 Spread Quality Distribution

Average spread quality (definition 3.1.6) from all quotes on DAX options from MM1, MM2 and MM3 (dataset 2).

Summary

Even though the spread increases with increasing bid price, the quality of the spread increases as well. This only changes for options in maturity category 5, where the spread is normally bigger than half the maximum spread.

CHAPTER 3.6 Factor Analysis of Market Maker Spreads

3.6.1 Linear Regression

After investigating the possible influence factors on spread we now use a linear regression method to verify or falsify our hypothesis.[1] As explanatory variables we will consider the four possible influence factors identified on page 84, which were the holding risk, bid price and two dummy variables, which test if the maximum spread is capped or floored.

As we have seen in chapter 3.3 (p. 85) a possible measure for the holding risk is squared delta, even though from a mathematically point of view, this seems to be a peculiar way of measuring risk. However, our observations in chapter 3.6 (p. 98) justify this approach by proving that squared delta seems to have a strong impact on spreads.
On page 86 we identified the time to expiry as a possible, simplified way to handle the time till offsetting trade arrives, i.e. as a measure for liquidity. Our data analyses in the previous chapter showed that the time to expiry has an impact on spreads.
We also observed a strong correlation between the call spread and the spread of a put with the same strike and expiry. Therefore we also consider the put spread in our linear regression model.

Because of the very strong correlation between bid price and delta squared (70%) as well as time to expiry (80%), we ignore bid prices to avoid collinearity.

As expected on page 91 the regression model seems to capture the influence factors on spreads best for MM1 and MM2. As can be seen in table 3.6.1 MM1 and MM2 have got R-squared values of 0.81 and 0.92, respectively. In contrast MM3 only has a R-squared of 0.51.

Estimate (Data points)	R-Squared	Intercept (T- Value)	Squared Delta (T- Value)	Days to Expiry (T- Value)	Put Spread (T- Value)	Bid >= Cap (T- Value)	Bid <= Floor (T- Value)
MM1 (2231)	0.81	0.93 (9.35)	7.01 (18.12)	0.01 (69.01)	0.01 (4.26)	-0.77 (-8.29)	NA
MM2 (2687)	0.92	-0.55 (-8.88)	2.57 (17.09)	0.03 (9.30)	0.80 (56.74)	0.92 (12.45)	-1.32 (-10.10)
MM3 (5255)	0.51	-0.69 (-5.12)	10.81 (43.89)	0.01 (26.08)	0.30 (22.06)	-0.04 (-0.33)	NA

Table 3.6.1 Call Option Regression on 10/04/2006,

All three market values, i.e. the squared delta, time to expiry and the spread of an according put option have a strong impact on the call spread. However the value of the intercept for MM2 and MM3 should be positive, to have any

1. For a brief introduction into linear regression models, see appendix A.6.

economical meaning. The borders to maximum spread size do not have any relevant influence on the spread for MM1 and MM3. If we ignore them the same picture arises from another linear regression. The main influence factor is different for each market maker, though (table 3.6.2). The main influence factor for MM1 is the time to expiry, with a correlation of 87% to the spread, and therefore giving an R-squared of 0.785. MM2 seems to adjust his call spreads mainly to the spread of a put option with same strike and expiry. For MM3 the squared delta seems to have the strongest impact on spreads.

Correlation	Bid	Squared Delta	Days To Expiry	Put Spread
MM1	0.87	0.62	0.87	0.06
MM2	0.88	0.05	0.90	0.93
MM3	0.59	0.44	0.47	0.20

Table 3.6.2 Spread to explanatory variables on 10/04/2006

3.6.2 Residual Analysis

A residual analysis[1] shows that the residuals are without any pattern for MM3 and can be assumed to be normal distributed for MM1 and MM3, see figure A.9 and A.10, respectively. As seen before MM1 tends to increase spread during market opening and closing times, resulting in higher residuals at these times. The residual analysis for MM2 shows that they are not normal distributed. A Shapiro-Wilk normality test clearly rejects the normal distribution assumption. This also strikes out in a distribution plot (figure 3.6.3).

A Durbin-Watson test or an Auto-Correlation-Plot shows that residuals for MM1 and MM3 are significantly autocorrelated. This suggests that changes in the intraday spread size, are influenced by generally unobservable market makers' specifications. One example are the jumps in spread size from MM3, see footnote 2, p. 91.

Because MM2 maintains an almost constant spread size for each option series (figure 3.5.5), a linear regression over one day seems to be unable to measure the influence from the explanatory variables.

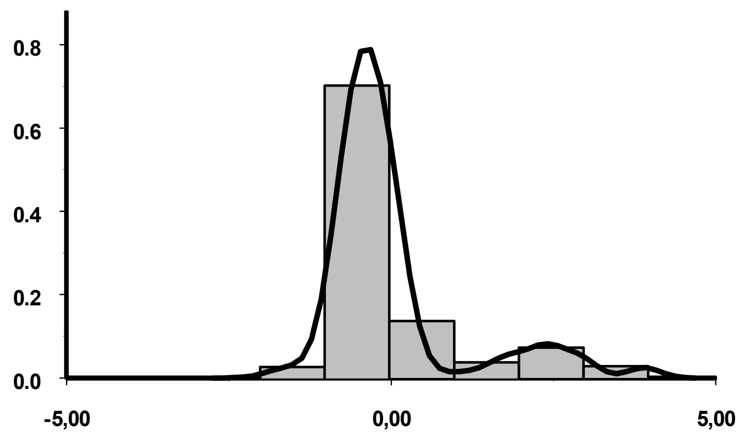


Figure 3.6.3 MM2 Residual Distribution

Residuals from a linear regression (see table 3.6.1) on MM2 spreads.

1. A residual analysis is necessary, because it is possible to get a high R^2 and t-statistics that appear to be significant, without any meaningful interpretation. See Enders [35], pp.170-175.

CHAPTER 3.7 Summary

3.7.1 Results

In this chapter we analysed the cross-sectional distribution of four different market maker spreads on DAX, SMI and ESX options. We observed significant differences between individual market makers, for example in the intraday volatility of spreads, from very volatile (MM3) to almost constant (MM2). We also saw that we can divide market makers into competitive or aggressive market makers (MM1, MM2 and MM3), fighting for the flow by offering tight spreads, and non-aggressive market makers (MM4), who just try to maintain maximum allowed spreads.

A lot of similarities between the quoting behaviour of competitive market makers could be observed. We showed that the defining attributes of the quoted option (option type, moneyness and time to expiry), which are directly correlated to the liquidity, have a big impact on market makers' spreads. In general, option series which are liquid have a lower spread than illiquid options.[1] We also derived that the underlying risk of holding an open position is directly correlated to the size of the spread. One way to measure the risk for a call (put) option would be to price a call on call (put) option. However, as we have proven in the theoretical part (chapter 2.2.6), the compound option price in the Black-Scholes model is correlated to the Black-Scholes delta of the underlying option. A strong correlation between the squared delta and the spread size could be found. We also realized a strong correlation between call and put options with the same strike and time to expiry. A portfolio consisting of a long call, a short put and one unit of stock is risk free. A short put can hence be thought of as a substitute for a short call option, with the aim to nullify the risk involved in holding an open long call contract. On the other hand for short term options the spread of a call option with strike $S \cdot K$ (where S is the price of the underlying) was similar to the spread of a put option with strike $S^{-1} \cdot K$. We could explain this with a spread following delta. However, note that this implies that the market crash phobia (Hull [55]), only has impact on the implied volatility smiles (as seen in chapter 2.3.2) and not on the spread size.

Apparently, only easily accessible market variables have any impact on the spread. In contrast, neither correlation between spread quality and the volatility of VDAX-NEW nor the volatility of historical volatility could be found.

The results show that the theoretical spread, derived in chapter 2.5, does not model observed spreads. The theoretical and observed spreads just have in common that the absolute spread increases by time to expiry and that the relative spread increases by out of the moneyness, and therefore the spread quality decreases by out of the moneyness. One big difference is that the theoretical spread is highest

1. For example, the average spread on put options on the DAX is lower by about 0.5 index points than from call options. During the days investigated the average put/call DAX ratio was about 1.2. But a very high liquidity can also result in higher spreads, for example the put/call ESX ratio on these days was about 1.61, but the spread on put options was higher by about 0.3 index points.

for ATM options, but the observed spread is highest for ITM options. Note, however, if one considers super-replicating strategies the maximum is generally found for ITM options.[1]

3.7.2 Problems

Instead of considering traded or best bid/ask quotes, we used actual market maker quotes, which induces the problem that model specific variables can bias the results. These variables may be defined by the exchange, like maximum spreads which completely determined spreads from MM4, or they may originate from market makers, like the high intraday volatility of spreads from MM3. If one wants to analyse market maker quotes, e.g. to know how each one participates, it is therefore important to be able to identify and ignore market makers like MM4. This also suggests to consider intraday data, because otherwise model specific originalities as observed by MM3 could not be revealed and taken into consideration. However, the vast amount of data – more than 200,000 valid quotes for DAX options in dataset 3 – restricted this analysis to a small time span and a few market makers. Out of this reason the linear regression model was only calculated for each day separately.

3.7.3 Outlook

On page 95 we pointed out that it would be promising to further analyse the connection between spread size and the implied volatility surface. Stochastic volatility models induce a relationship between the implied volatility surface and the volatility of volatility[2]. Instead of trying to approximate the volatility of volatility from market data, we could therefore try to find a relationship between the spread size and the convexity of the smile curve. But because a single market maker does not tend to include these variables, the analysis should be done on the whole market, meaning only actually traded quotes.
Also some improvements to the analysis performed would be favourable. Especially a compound option, priced in a stochastic volatility model, would capture the risk involved much more efficiently than using squared delta as a risk measure. The time till offsetting trade should be measured with a random variable, which would have to be calibrated to market data.

An analysis on newly introduced weekly options (24 April 2006) should verify or falsify some hypotheses made in this chapter. Especially the increasing spread for ITM options, just before expiry, should be visible for weekly options. At the moment the market seems to be not liquid enough to make this study; only between 40,000 and 50,000 weekly option contracts were traded from inception till 7 July 2006 [42].

Another approach to study market makers quoting behaviour would be to emphasise the specific options that are quoted. An overview of what kinds of options were quoted in dataset 3 and 2 can be seen in tables A.4 and A.5, respectively. For example, MM1 prefers to quote OTM options, i.e. calls with high strikes and puts with low strikes, whereas MM4 quotes the same strikes for call and put options.

1. See chapter 2.5 and the references cited there.
2. Namely that the convexity of the smile is regulated by the volatility of volatility, see figure 2.4.1.

Appendix

Fundamental Mathematical Theorems

A.1 Itô Formula

If we have a continuous, d-dimensional semimartingale $X_t = (X_{1,t}, \ldots, X_{d,t})$ and if $f : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}$ is in $\mathcal{C}^{1,2}$, i.e. $f(t, x)$ is once continuously differentiable in t and twice in x. Then we have:

$$f(t, X_t) = f(0, X_0) + \int_0^t \frac{\partial}{\partial t} f(s, X_s) ds + \sum_{i=1}^{d} \int_0^t \frac{\partial}{\partial x_i} f(s, X_s) dX_{i,s}$$
$$+ \frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} \int_0^t \frac{\partial^2}{\partial x_i \partial x_j} f(s, X_s) d\langle X_i, X_j \rangle_s$$

Proof: See Karatzas and Shreve [60], p. 153. q.e.d.

A.2 Girsanov Theorem

Let $W_t = (W_{1,t}, \ldots, W_{n,t})$ be an n-dimensional $\mathbf{P}$-Brownian motion and let $\mathbf{F} = (\mathcal{F}_t)$ be a filtration, satisfying the usual conditions. Let $X_t = (X_{1,t}, \ldots, X_{n,t})$ be an n-dimensional, $\mathbf{F}$-predictable process, so that $L_t := \mathcal{E}(X \bullet W)_t$ (definition 2.1.3) is a martingale (for example X_t satisfies Novikov's Condition). Then we can define a new probability measure $\mathbf{Q}$ by:

$$\mathbf{Q}(A) = \int_A L_t d\mathbf{P} \text{ for all } A \in \mathcal{F}_t \qquad \text{(EQ 1)}$$

and

$$W_t^* = W_t + \int_0^t X_s ds$$

is an n-dimensional $\mathbf{Q}$-Brownian motion.

Proof: See, for example, Karatzas and Shreve [60], pages 191-195. q.e.d.

A.3 Martingale Representation Theorem

Let $W_t = (W_{1,t}, \ldots, W_{n,t})$ be an n-dimensional $\mathbf{P}$-Brownian motion and let $\mathbf{F}$ be the augmentation of the filtration generated by the Brownian motion W_t.
For any $(\mathbf{P}, \mathbf{F})$-martingale N_t exists a uniquely defined n-dimensional, $\mathbf{F}$-adapted process $X_t = (X_{1,t}, \ldots, X_{n,t})$ with:

$$N_t = N_0 + \sum_{k=1}^{n} \int_o^t X_{k,s} dW_{k,s}$$

Proof: See Björk [10], page 157. q.e.d.

A.4 Generalized Itô-Venttsel Formula

Let $W = (W_1, \ldots, W_m)$ be a m-dimensional standard Brownian motion on $\mathbb{R}^m$. Let $G(t, u)$ be twice differentiable with respect to u and satisfy the SDE

$$dG(t, u) = A_1(t, u)dt + B_1(t, u)dW_t$$

If u_t satisfies the SDE

$$du_t = A_2(t, u_t)dt + B_2(t, u_t)dW_t$$

Then we can combine both SDE's to get a SDE for $G(t, u_t)$ given by:

$$\begin{aligned} dG(t, u_t) = {} & A_1(t, u_t)dt + B_1(t, u_t)dW_t \\ & + \frac{\partial G}{\partial u}(t, u_t)du_t + \frac{1}{2}\frac{\partial^2 G}{\partial u^2}(t, u_t)|B_2(t, u_t)|^2 dt \\ & + \frac{\partial B_1}{\partial u}(t, u_t)B_2(t, u_t)dt \end{aligned}$$

Proof: See Venttsel [81], quoted from Hafner [51]. q.e.d.

Data Handling

A.5 Correlation

The correlation coefficient between two observed variables $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$ is a measurement for the quality of a least square fitting.
Let b be the least square fit for

$$y = xb + a \qquad \text{(EQ 2)}$$

with $b \in \mathbb{R}$ and $a \in \mathbb{R}^n$. We have:

$$b = \frac{COV(x, y)}{VAR(x)} = \frac{\sum_{i=0}^{n} (x_i - \mathbb{E}[x])(y_i - \mathbb{E}[y])}{\sum_{i=0}^{n} (x_i - \mathbb{E}[x])^2}$$

Formula A.1 Least Square Fit

And let b' be the least square fit for $x = yb' + a'$, then the correlation coefficient r is defined to be:

$$r := \begin{cases} -\sqrt{bb'} & bb' \le 0 \\ \sqrt{bb'} & bb' \ge 0 \end{cases}$$

Formula A.2 Correlation Coefficient

Using $COV(x, y)^2 \le VAR(x)VAR(y)$ (Cauchy, Schwarz) we can easily see that $r \in [-1, 1]$ must be satisfied.

A.6 Linear Regression

If we allow x to be a $n \times m$ matrix in (EQ 2) and b an m-dimensional vector, i.e. each observed sample y_i can be explained by m different variables $(x_{i1}, \ldots, x_{im})$, we have a linear regression model. In which we will write (EQ 2) as:

$$y = \mathbf{X}b + a \qquad \text{(EQ 3)}$$

with $b \in \mathbb{R}^m$, $a \in \mathbb{R}^n$ and $\mathbf{X} = \begin{bmatrix} x_1 \\ \ldots \\ x_n \end{bmatrix} = \begin{bmatrix} x_{11} & \ldots & x_{1m} \\ \ldots & \ldots & \ldots \\ x_{n1} & \ldots & x_{nm} \end{bmatrix}$

As before we are interested in an estimate $b \in \mathbb{R}^m$ that minimizes the residual sum of squares (RSS):

$$RSS := \sum_{i=1}^{n} (y_i - x_i^t b)^2$$

It can be shown that the solution is given by:[1]

$$b = (\mathbf{X}^t\mathbf{X})^{-1}\mathbf{X}^t y$$

1. See e.g. Hamilton [52], p.202.

The quality of the estimation is given by the centered R^2, defined by:

$$R^2 := \frac{y^t\mathbf{X}(\mathbf{X}^t\mathbf{X})^{-1}\mathbf{X}^t y - n\mathbf{IE}[y]^2}{y^t y - n\mathbf{IE}[y]^2}$$

Formula A.3 Centered R squared

In general residuals are assumed to be independent and normal distributed with zero mean. Because this implies, that the estimate b is optimal and independent to the residuals.[1]

Data Details

A.7 DAX

Table A.4 Count of Data Points

Table A.4 shows the total number of 10-minute intervals over all eight different days in which each market maker quoted options of the different series on DAX options (dataset 3).

Option Type	Expiry	Moneyness	MM1	MM2	MM3	MM4	Grand Total
Call	1	ITM	711	3343	1713	1495	7262
		ATM	895	1063	778	831	3567
		OTM	1114	1585	461	682	3842
	1 Total		**2720**	**5991**	**2952**	**3008**	**14671**
	2	ITM	1194	3115	5109	2771	12189
		ATM	1481	1573	1430	1377	5861
		OTM	2246	4641	3661	787	11335
	2 Total		**4921**	**9329**	**10200**	**4935**	**29385**
	3	ITM	682	621	4522	1229	7054
		ATM	850	776	823	866	3315
		OTM	1027	1476	3491	512	6506
	3 Total		**2559**	**2873**	**8836**	**2607**	**16875**
	4	ITM	680	612	3528	1466	6286
		ATM	845	582	824	716	2967
		OTM	1019	1628	3930	413	6990
	4 Total		**2544**	**2822**	**8282**	**2595**	**16243**
	5	ITM	2229	729	3520	3454	9932
		ATM	2024	1408	1450	1790	6672
		OTM	3328	5053	7624	2529	18534
	5 Total		**7581**	**7190**	**12594**	**7773**	**35138**
Call Total			**20325**	**28205**	**42864**	**20918**	**112312**

1. See Hamilton [52], p.204.

Option Type	Expiry	Moneyness	MM1	MM2	MM3	MM4	Grand Total
Put	1	ITM	743	1727	1200	1017	4687
		ATM	824	1060	741	835	3460
		OTM	1944	3892	664	1271	7771
	1 Total		**3511**	**6679**	**2605**	**3123**	**15918**
	2	ITM	1188	1732	4473	806	8199
		ATM	1481	1572	1429	1368	5850
		OTM	2675	6492	4544	2707	16418
	2 Total		**5344**	**9796**	**10446**	**4881**	**30467**
	3	ITM	659	555	3632	517	5363
		ATM	858	831	823	863	3375
		OTM	1162	1383	4192	1217	7954
	3 Total		**2679**	**2769**	**8647**	**2597**	**16692**
	4	ITM	655	707	4127	415	5904
		ATM	849	795	822	716	3182
		OTM	1107	1397	3013	1455	6972
	4 Total		**2611**	**2899**	**7962**	**2586**	**16058**
	5	ITM	2239	3383	7782	2527	15931
		ATM	2036	1725	1407	1797	6965
		OTM	3357	2362	2117	3445	11281
	5 Total		**7632**	**7470**	**11306**	**7769**	**34177**
Put Total			**21777**	**29613**	**40966**	**20956**	**113312**
Grand Total			**42102**	**57818**	**83830**	**41874**	**225624**

Table A.5 Count of Data Points

Table A.5 shows the total number of valid, quoted options of the different series on DAX options summed up for each day during the whole time available (dataset 2).

Option Type	Expiry	Moneyness	MM1	MM2	MM3	MM4	Grand Total
Call	1	ITM	368	515	480	356	1719
		ATM	358	430	338	284	1410
		OTM	478	675	348	362	1863
	1 Total		**1204**	**1620**	**1166**	**1002**	**4992**
	2	ITM	778	961	1050	718	3507
		ATM	734	734	734	562	2764
		OTM	1034	1159	1052	664	3909
	2 Total		**2546**	**2854**	**2836**	**1944**	**10180**
	3	ITM	380	248	516	366	1510
		ATM	362	306	362	314	1344
		OTM	502	438	512	402	1854
	3 Total		**1244**	**992**	**1390**	**1082**	**4708**
	4	ITM	382	262	516	372	1532
		ATM	362	294	362	304	1322
		OTM	502	428	512	392	1834
	4 Total		**1246**	**984**	**1390**	**1068**	**4688**
	5	ITM	1226	566	1050	1192	4034
		ATM	838	540	754	720	2852
		OTM	1548	1060	1373	1276	5257
	5 Total		**3612**	**2166**	**3177**	**3188**	**12143**
Call Total			**9852**	**8616**	**9959**	**8284**	**36711**

Option Type	Expiry	Moneyness	MM1	MM2	MM3	MM4	Grand Total
Put Total	1	ITM	384	532	484	368	1768
		ATM	360	429	342	284	1415
		OTM	488	713	394	356	1951
	1 Total		**1232**	**1674**	**1220**	**1008**	**5134**
	2	ITM	784	981	1052	664	3481
		ATM	734	752	734	562	2782
		OTM	1006	1329	1050	716	4101
	2 Total		**2524**	**3062**	**2836**	**1942**	**10364**
	3	ITM	382	376	512	402	1672
		ATM	362	338	362	314	1376
		OTM	488	374	516	366	1744
	3 Total		**1232**	**1088**	**1390**	**1082**	**4792**
	4	ITM	382	382	512	392	1668
		ATM	362	320	362	304	1348
		OTM	488	356	514	372	1730
	4 Total		**1232**	**1058**	**1388**	**1068**	**4746**
	5	ITM	1224	964	1273	1276	4737
		ATM	838	676	682	720	2916
		OTM	1568	902	926	1192	4588
	5 Total		**3630**	**2542**	**2881**	**3188**	**12241**
Put Total			**9850**	**9424**	**9715**	**8288**	**37277**
Grand Total			**19702**	**18040**	**19674**	**16572**	**73988**

Figure A.6 Verification: Calculation of VDAX-NEW

Verification of published VDAX-NEW and 30-day VDAX-NEW from own calculations:

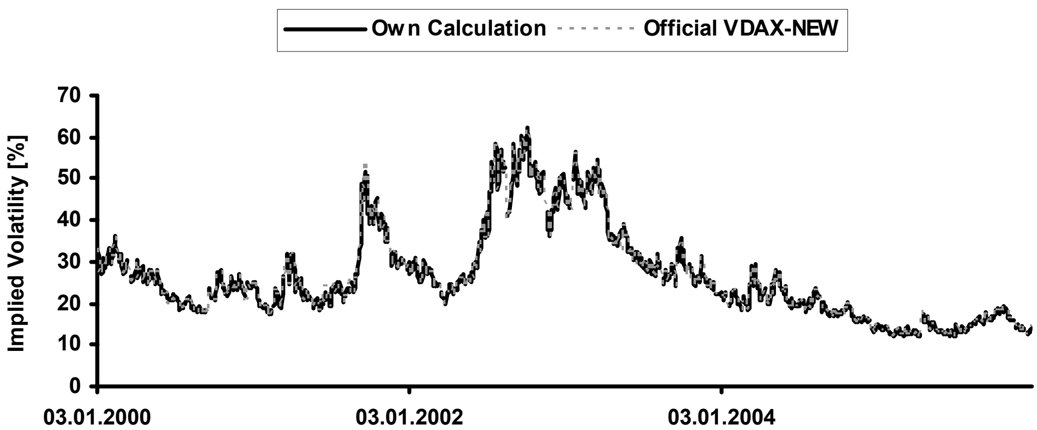


Figure A.7 Spread Versus Quote Size

Cross-sectional distribution of spread size, by expiry category (1 to 4), moneyness and quote size 1 (quotes with size less than 100), 2 (less than 300) and 3 (else) from MM1, MM2, MM3 on all available days from DAX options (dataset 3).

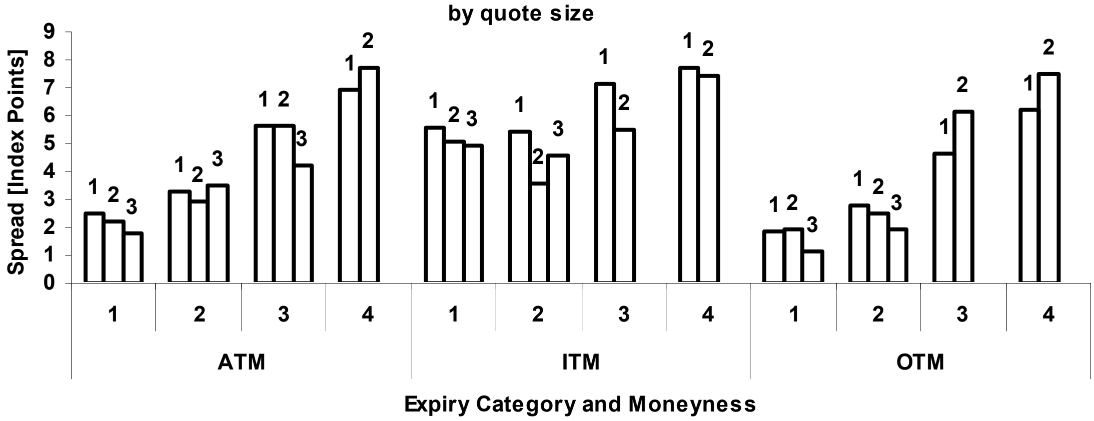


Figure A.8 Verification: Spread Increases While Reaching Expiry

MM1 spread quality from all DAX options expiring in April (black dotted), May (grey dotted), June (black solid), July (grey solid) and August (grey thin) (dataset 2).

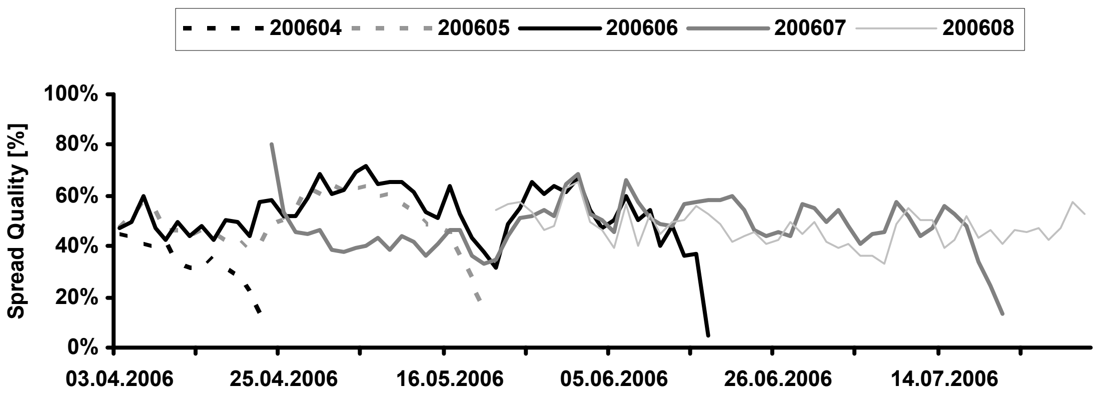


Figure A.9 MM1 Residual Distribution And QQ-Normal Plot

Residuals from a linear regression (see table 3.6.1) on MM1 spreads.

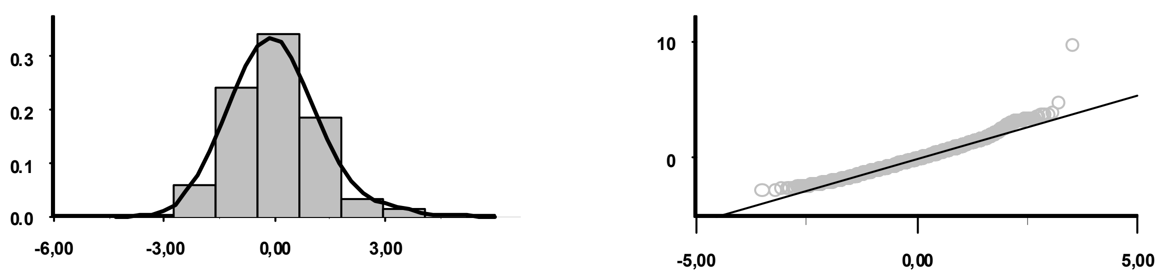
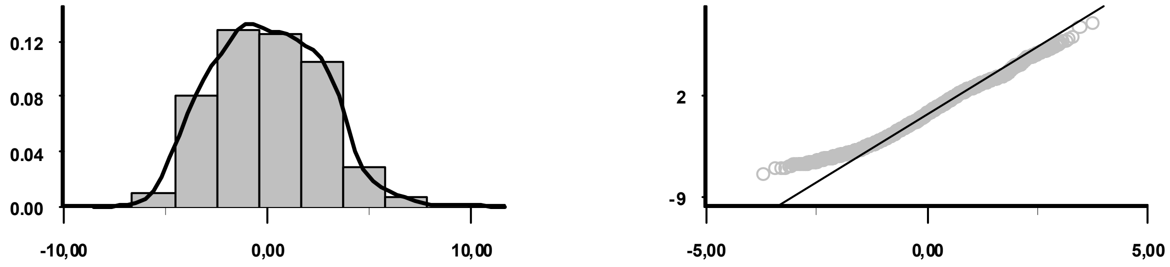


Figure A.10 MM3 Residual Distribution And QQ-Normal Plot

Residuals from a linear regression (see table 3.6.1) on MM3 spreads.



A.8 ESX

Figure A.11 Spread Distribution (ESX)

Average Spread from all quotes on ESX options from MM1, MM2 and MM3 (dataset 3).

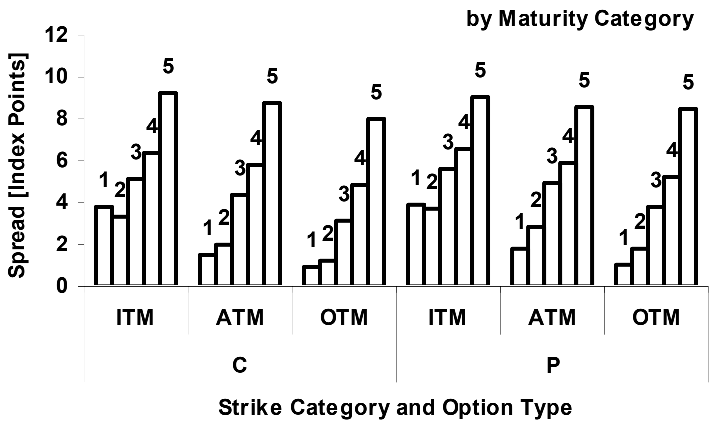
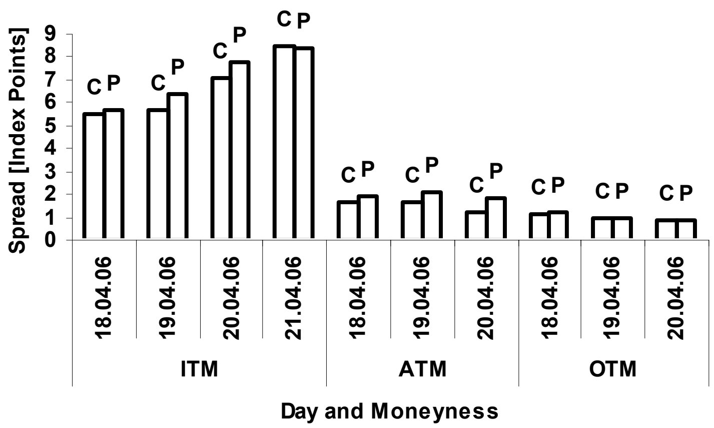


Figure A.12 Spreads Before Expiry (ESX)

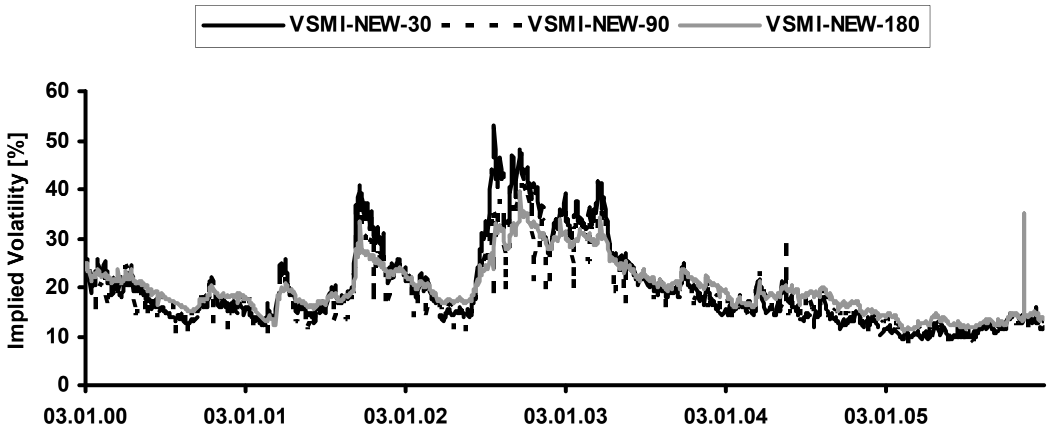
Average spread from ESX options expiring on 21.04.2006 by option type call (C) and put (P) from MM1 and MM3 (dataset 3).



A.9 SMI

Figure A.13 VSMI-NEW

Long term (grey) and short term (black) VSMI-NEW comparison.



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