Goethe University, Frankfurt/Main

Thesis

Construction Of the Implied Volatility Smile

by

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Even extremely liquid markets, as the market for European style options on equity indexes, sometimes fail to provide sufficient data for pricing its options, e.g. particular options are not liquid enough.

We are to investigate an extension of a well-known and widely spread "market-based" Vanna-Volga method, which not only allows to retrieve reasonable estimates for option premiums, but also to determine consistent implied volatilities easily. The theoretical results are then analyzed using the daily settlement prices of Dow Jones EURO STOXX 50 call options provided by Deutsche Börse. Introducing a stochastic volatility model we were also able to deliver an explanation for the formulas, which were previously heuristically justified merely by formal expansion of the option premium by Itô.

Acknowledgements

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Great thanks to my parents for their backing.

1 Introduction

1.1 Motivation

We investigate a method for a simple and transparent derivation of the implied volatility smile from the market data. A knowledge of the contemporary volatility is crucial for traders, especially in Foreign Exchange market, since options are priced in terms of volatility. But also traders in other markets are interested in an easily reproducible methodology to retrieve the volatility smile – for hedging, exploiting arbitrage or trading volatility spreads. The procedure also delivers options premiums. This can be used for pricing illiquid options, e.g for deep in-the-money options. The investigated procedure requires the existence of four liquid options, whose implied volatilities are then readily available. By adjusting the theoretical Black-Scholes price with costs for an over-hedge one receives the desired market consistent option premium.

An Ornstein-Uhlenbeck stochastic volatility process provides the theoretical framework. Being mean-reverting, the volatility process tends to its mean level. Thus, experiencing a fast mean-reverting volatility, we are able to approximate option premiums by the Black-Scholes price adjusted by higher order derivatives of the option premium with respect to the underlying spot price.

The theoretical results are evaluated with options on a highly traded Pan-European index DJ EURO STOXX 50. The settlement data were provided by Deutsche Börse.

1.2 Outline

A brief overview of Eurex is given in Chapter 2.
Mathematical and economical terms referred to in this thesis are dealt with in Chapter 3.
In Chapter 4 we use the Vanna-Volga method for deriving option premiums and volatility smile.
Chapter 5 deals with an extension of the Vanna-Volga method. Obtained results are compared in Chapter 6.

Pricing under mean-reverting stochastic volatility together with the resulting pricing formula is explained in Chapter 7. The Appendix contains auxiliary tables and figures as well as a description of the Matlab program.

The set of data for EURO STOXX 50 provided by Microstrategy* contains:

• Underlying close price	S_t
• Call and put settlement prices	C_t^{MK} P_t^{MK}
• Strike price	K
• Time to expiration	$\tau = T - t$
• Implied volatility	I

We restrain our observations on options with *fixed-strike moneyness* $\mathfrak{m} := \frac{K}{S}$

$$0.8 < \mathfrak{m} < 1.2 \,,$$

since they are the most liquid in the market and thus bear the most information. The other restriction is the consideration of call options only - the same results can be obtained for put options due to the put-call parity (see Definition 5). Put options are used for the estimation of the interest rate.

*Data portal by Deutsche Börse

2 Eurex

Eurex is the world leading derivatives exchange. It offers fully electronic trading in a large number of derivatives. Facilities like Wholesale Trading (OTC), Eurex Strategy Wizard or Market Making are available to the market to ensure liquidity and simplicity in trading. We take a look at derivatives on the EURO STOXX 50 traded at Eurex relevant for the thesis. A brief introduction of the Market Making Program thereafter is connected to the question stated in the Chapter 5.2.

2.1 Derivatives on DJ EURO STOXX 50 at Eurex

Dow Jones EURO STOXX 50 is a blue-chip index containing the top 50 stocks in the Eurozone.* The stocks, capped at ten percent, are weighted according to their free float market capitalization with prices updated every 15 seconds. DJ EURO STOXX 50 Index derivatives are the world's leading euro-denominated equity index derivatives.

European style options on the Index traded at Eurex (OESX) are available with maturities up to 10 years. The last trading day is the third Friday of each expiration month. The final settlement price is calculated as the average of the DJ EURO STOXX 50 Index values between 11:50 and 12:00 CET. The contract value is EUR 10. The minimum price change is 0.1 index point which is equivalent to EUR 1. At least seven exercise prices are available for each maturity with a term of up to 24 months. For these maturities the exercise price intervals are 50 index points; The exercise price interval for options with maturities larger than 36 month is 100 index points.

Futures on the EURO STOXX 50 (FESX) have excellent liquidity having a minimum price change of one index point which is equivalent to EUR 10.

VSTOXX, based on the DJ EURO STOXX 50 options, is an implied volatility index traded on the Deutsche Börse. It is set up as a rolling index with 30 days to expiration and derived by linear interpolation of the two nearest sub-indexes.† Sub-indexes per

*For current composition of the index visit www.stoxx.com .

†For detailed information on the derivation see [11].

option expiry are computed for the first 24 months, giving 8 sub-indexes in total, which are updated once a minute.

Volatility futures on VSTOXX (FVSX) can be used for trading calendar and market spreads, as a hedging tool (e.g. crash risk), speculating (e.g mean-reverting nature of volatility).

2.2 Market Making at Eurex

Designated traders, called *Market Makers*, are granted a license to make tight markets in options and several futures contracts. This increases liquidity and transparency. Every exchange participant may apply to be a Market Maker. Three models, which differ in the kind of response to quote requests, continuous quotation and products selection, may be chosen.
Specifically:

- *Regular Market Making* (RMM) is restricted to less liquid options on equities, equity indexes and Exchange Traded Funds (EXTF) and to all options in fixed income (FX) futures. RMM allows to choose products to quote (if available) with the obligation to response to quote requests in all exercise prices and all expirations.

- *Permanent Market Making* (PMM) is available for all equity, equity index, EXTF options and options on FX futures. Products in which participants would like to act as PMM can be selected individually. The obligation is to quote for a set of pre-defined number of expirations 85% of the trading time continuously.

- *Advanced Market Making* (AMM) is available for any pre-defined package of equity and/or equity index options as well as on FX options with an obligation of continuous quotation for a set of exercise prices for a pre-defined number of expirations and options.

If traders fulfill the obligations they are refunded transactions and exercise fees.

We give here a detailed description of PMM in index options only.[‡] PMM consists of three obligation levels – PMM, PMM short (PMS) and PMM long (PML). The obligation

[‡]For further information on Market Making please visit www.eurexchange.com > Market Model > Market-Making.

to quote for an average of 85 percent must be fulfilled for all expirations up to a defined maximum maturity. Asymmetric quotation is allowed. PMM and PMS are obliged to quote calls and puts in five exercise prices out of a window of seven around the current underlying price – one at-the-money, three in-the-money and three out-of-the-money exercise prices. Compared to PMM, PMS must quote a larger minimum quote size, but fewer expirations.
PML concentrates on long-term expirations – more than 18 and up to 60 months. The obligations are fulfilled by quoting six exercise prices out of a window of nine around the current underlying price. For that compare Figure 2.1 where the contract months up to 10 years are defined as follows:

> The three nearest successive calendar months (1-3), the three following quarterly months of the March, June, September and December cycle (6-12), the four following semi-annual months of the June and December cycle (18-36) and the seven following annual months of the December cycle (48-120).

A protection tool against system-based risk, called *"Market Maker Protection"*, is provided by Eurex for Market Makers in PMM and AMM. It averts too many simultaneous trade executions on quotes by Market Maker by counting the number of traded contracts per product within a defined time interval, chosen by the Market Maker.

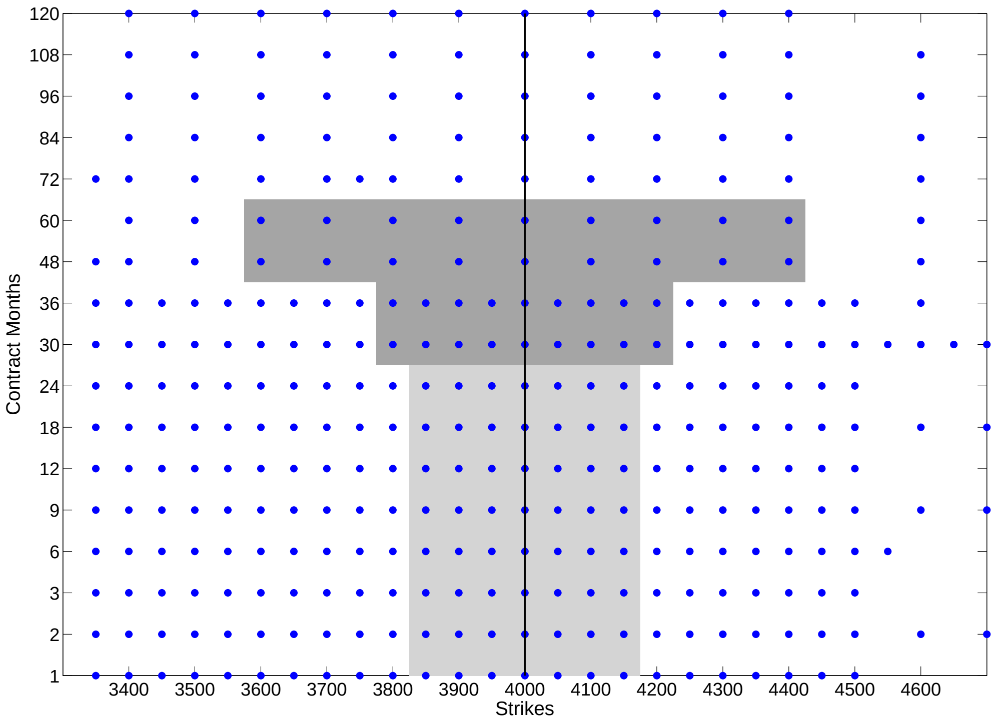

Figure 2.1: Trading obligations for PMM and PML.
The solid vertical line denotes the at-the-money strike. The light grey area is the PMM area with the obligation to quote five out of seven strikes. The dark grey is the PML area with the obligation to quote six out of nine strikes
.

3 Preliminaries

3.1 Mathematical Terms

We first introduce some theorems and definitions.

Let $T^* > 0$ be a finite time horizon. We consider a complete filtered probability space $(\Omega, \mathcal{F}, \mathbf{P}, \mathbf{F})$ satisfying the usual conditions,* where Ω is the set of states, $\mathcal{F}$ is a σ-algebra, $\mathbf{P} : \mathcal{F} \to [0,1]$ a probability measure and $\mathbf{F} := (F_t)_{t\in[0,T]}$ a filtration generated by a n-dimensional Brownian motion W_t.

We present a generalized version of Girsanov's Theorem.

Theorem 1 (Girsanov's Theorem)
Let X_t be a process with

$$X_t - X_0 = \int_0^t \mu_u du + \int_0^t \sigma_u dW_u \tag{3.1}$$

where $\mu : \Omega \times \mathbb{R}_+ \to \mathbb{R}$ and $\sigma : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ are adapted and $(\mathcal{F} \otimes \mathbb{R}_+)$-measurable, with

$$\int_0^T \sigma_u^2 du < \infty \quad \mathbf{P}\text{-a.s.} \qquad \text{and} \qquad \int_0^T |\mu_u| du < \infty \;\; \mathbf{P}\text{-a.s.}$$

Let r_t be an adapted process with $\int_0^T |r_u| du < \infty$ ($\mathbf{P}$-a.s.) such that

$$\int_0^T \left(\frac{\mu_u - r_u}{\sigma_u}\right)^2 du < \infty \, .$$

We set

$$Z_t = \exp\left(-\int_0^t \frac{\mu_u - r_u}{\sigma_u} dW_u - \frac{1}{2}\int_0^t \left(\frac{\mu_u - r_u}{\sigma_u}\right)^2 du\right) \, .$$

If $\mathbf{E}\big[Z_T\big] = 1$ then we can define a new measure $\mathbf{Q}$ such that

$$\frac{d\mathbf{Q}}{d\mathbf{P}} = Z_T \, ,$$

*Right continuous and saturated for $\mathbf{P}$-null sets.

then X has the representation under $\mathbf{Q}$

$$X_t - X_0 = \int_0^t r_u du + \int_0^t \sigma_u d\hat{W}_u \ ,$$

where $\hat{W}_t$ is a $\mathbf{Q}$ Brownian motion.

Proof

See [18], Chapter VII, §3b. ∎

A process σ_t is called *square integrable*, denoted by $\sigma_t \in \mathcal{L}^2$, if

$$\int_0^T \mathbf{E}\big[\sigma_u^2\big] du < \infty \ .$$

Given a process X_t, processes

$$t \mapsto \mu(t, X_t) \qquad t \mapsto \sigma(t, X_t) \qquad t \mapsto r(t, X_t)$$

also satisfy Girsanov's theorem.

A process of the form (3.1) is referred to as an *Itô process.* Girsanov's Theorem allows a representation of Itô processes with respect to a shifted Brownian motion $\hat{W}_t$, which naturally defines a new measure, an *equivalent martingale measure* – measure, giving probability zero to events, which had probability zero under the initial measure.

The first integral in (3.1) is the Riemann-Stieltjes integral, the second is a *stochastic integral* defined to be an $\mathcal{L}^2$-limit of an approximating sequence of *simple*† functions :

$$\lim_{n\to\infty} \int_a^b g_n(u) dW_u = \int_a^b g(u) dW_u := \sum_{k=0}^{n-1} g(t_k)\big[W_{t_{k+1}} - W_{t_k}\big] \ , \tag{3.2}$$

where $\int_a^b \mathbf{E}\big[(g_n(u) - g(u))^2\big] du \to 0$ and $a = t_0 < \cdots < t_n = b$.
The stochastic integral in (3.2) is evaluated with *forward increments* of the Brownian motion. This has an economic interpretation and is closely related to the point of arbitrage: interpreting g as the number of assets bought at t_k and held till t_{k+1} and W_t as the price of a driftless asset at t one profits $g(t_k)\big[W_{t_{k+1}} - W_{t_k}\big]$. If one would be able to anticipate the price evolution, a riskless profit could be possible.

Consider a portfolio consisting of $n + 1$ underlyings (without loss of generality we

†That is, there exist deterministic time points $a = t_0 < \cdots < t_n = b$, such that $\sigma(t, X_t)$ is constant on each subinterval.

assume the first underlying to be a riskless bond B_t)

$$V_t^\zeta = \zeta_t^0 B_t + \sum_{i=1}^n \zeta_t^i \cdot X_t^i \,.$$

The vector of adapted processes $\zeta_t = (\zeta_t^1, \dots, \zeta_t^n)$, with $\int_0^t |\zeta_u^0| du < \infty$, $\forall t \le T^*$ and ζ_t^i , $i = 1 \dots n$ is square integrable, is called a *trading strategy.* V_t^ζ is referred to as the *value of the portfolio* at time t.

The portfolio V^ζ described above is *self-financing* if its value vary only due to the variations of the market

$$V_t^\zeta - V_0^\zeta = \int_0^t \zeta_u^0 dB_u + \sum_{i=1}^n \int_0^t \zeta_u^i \cdot dX_u^i \,.$$

Definition 1 (Arbitrage Opportunity)
An *arbitrage opportunity* is a self-financing portfolio ζ such that

$$\begin{aligned} V_0^\zeta &= 0 \,, \\ V_T^\zeta &> 0 \,, \quad \mathbf{P}\text{-a.s.} \end{aligned}$$

The subsequent result by Delbaen and Schachermayer links the existence of an equivalent martingale measure stated by Girsanov's Theorem with the absence of arbitrage opportunities.

Theorem 2 (Fundamental Theorem of Asset Pricing)
There exists an equivalent martingale measure for the market model if and only if the market satisfies the NFLVR ("no free lunch with vanishing risk") condition.

Proof
See [4]. ■

A financial market is called *complete* if every contingent claim H with maturity T can be replicated by trading a self-financing strategy ζ, that is the value of the portfolio held according to the trading strategy at time T equals the contingent claim

$$V_T^\zeta = H \quad \mathbf{P}\text{-a.s.}$$

Theorem 3 (Complete Market Theorem)
A financial market is complete if and only if there exists exactly one equivalent martingale measure.

Proof
See [8]. ∎

The following theorem describes the evolution of a *continuous semimartingale*, a process X_t that can be written as $M_t + A_t$, where M_t is a continuous local martingale and A_t is a continuous adapted process of bounded variation. This decomposition, called the *Doob-Meyer decomposition*, is unique for $M_0 = A_0 = 0$.[‡] Thus an Itô process is a continuous semimartingale, whose finite variation and local martingale parts (as those on the right-hand side of (3.1) respectively), satisfy that both $\int_0^t \mu(u, X_u)du$ and $\langle \int_0^t \sigma(u, X_u)dW_u, \int_0^t \sigma(u, X_u)dW_u \rangle$ are absolutely continuous. The most general form of a stochastic integral can be defined with a previsible bounded process as the integrand and a semimartingale as an integrator.

Theorem 4 (Itô's Lemma)
Let $f : \mathbb{R}^n \to \mathbb{R}$ be a twice continuously differentiable function and let $X = (X^1, \ldots, X^n)$ be a continuous semimartingale in $\mathbb{R}^n$. Then for all $t \geq 0$ holds

$$f(X_t) - f(X_0) = \sum_{i=1}^{n} \int_0^t \frac{\partial f}{\partial x^i}(X_u)dX_u^i + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \int_0^t \frac{\partial^2 f}{\partial x^i \partial x^j}(X_u)d\langle X^i, X^j\rangle_u$$

Proof
See [16], IV.32, p. 60. ∎

Itô's Lemma gives us a tool for handling stochastic processes. Loosely speaking, it is the stochastic version of the chain rule in ordinary calculus.

The following theorem establishes a link between partial differential equations and stochastic processes and thus is a formula for valuating claims.

Theorem 5 (Feynman-Kač Stochastic Representation Formula)
Assume that f is a solution to the boundary value problem

$$\frac{\partial f}{\partial t}(t,x) + \mu(t,x)\frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2(t,x)\frac{\partial^2 f}{\partial x^2}(t,x) = 0 \,,$$
$$f(T,x) = h(x)$$

[‡]A detailed discussion on martingales can be found in [6], Section 2.

where the process $\sigma(u, X_u)\frac{\partial f}{\partial x}(u, X_u)$ is square integrable and X defined as (for $s \geq t$)

$$X_s - X_t = \int_t^s \mu(u, X_u)du + \int_t^s \sigma(u, X_u)dW_u$$
$$X_t = x .$$

Then f has the representation

$$f(t, x) = \mathbf{E}\big[h(X_T)|\mathcal{F}_t\big] ,$$

where $\mathcal{F}_t$ is generated by a Brownian motion W_t.

Proof
See [1], Chapter 4, p. 59. ■

Definition 2 (Novikov's Condition)
A process φ satisfies Novikov's condition, if

$$\mathbf{E}\exp\left(\int_0^t \frac{1}{2}\varphi_u^2 du\right) < \infty .$$

3.2 Economic Terms

Definition 3 (European Option)
A contract, giving its holder the right, not the obligation, to buy one unit of a pre-defined asset, the underlying S, at a predetermined strike price K on the pre-defined date, the maturity date T, is called a *European call option.* Its payoff is

$$h(S_T) = (S_T - K)^+ .$$

A *European put option* gives its holder the right to sell one unit of the underlying. Its payoff is

$$h(S_T) = (K - S_T)^+ .$$

Definition 4 (Forward)
A *forward contract* is an agreement between two parties to buy (sell) one unit of an underlying for a predefined price, the forward price, on a maturity date.

Definition 5 (Put-Call Parity)
The *put-call parity* is a relationship linking the option premiums for a call and a put option with same maturity and strike price

$$C(t, S_t; K, T) - P(t, S_t; K, T) = S_t - \exp(-r\tau)K \ .$$

This relationship follows from no-arbitrage arguments and is model-independent.

3.3 The Black-Scholes Model

The Black-Scholes model was introduced 1973 and started a profound study of the theory of option pricing.

3.3.1 Model Assumptions

The assumptions (see [2]) of the Black-Scholes model reflecting ideal conditions in the market are summarized below:

1. The market is efficient, that is arbitrage-free, liquid, time-continuous and has a fair allocation of information. That implies zero transaction costs.
2. Constant risk-free rate r.
3. The no dividend paying underlying follows a geometric Brownian motion: a process described in (3.4).
4. Short selling are possible.

None of the assumptions is satisfied perfectly. Markets have transaction costs, underlyings do not follow a geometric Brownian motion and are traded in discrete units or at most in fractions.[§] Despite those inconsistencies it is still a benchmark for other models and a standard pricing model in the financial world, since it is a function of observable variables and is easily implemented having a closed pricing formula. The main distinguishing feature of the Black-Scholes model is its completeness.

[§]The distribution of returns appears to be *leptokurtic* – higher peak around its mean and fat tails, compared to the standard normal distribution.

3.3.2 Dynamics of the Underlying

A bank account B_t with deterministic continuously compounded interest rate r exists and an investment of $B_0 = 1$ evolves as:

$$B_t - 1 = \int_0^t rB_u du \tag{3.3}$$

equivalent to

$$B_t = \exp(rt)\ . \tag{3.3$'$}$$

The price of the underlying S_t follows a geometric Brownian motion:

$$S_t - S_0 = \int_0^t \mu S_u\, du + \int_0^t \sigma S_u\, dW_u\ , \tag{3.4}$$

where μ denotes the instantaneous expected return of the underlying, σ^2 is a non-stochastic instantaneous variance of the return and at most a known function of time, W_t is a Brownian motion.
This implies a lognormal distribution of the underlying. To see that apply Itô's lemma on $G := \ln S_t$

$$G_t - G_0 = \int_0^t (\mu - \frac{1}{2}\sigma^2) du + \int_0^t \sigma dW_u = (\mu - \frac{1}{2}\sigma^2)t + \sigma W_t.$$

That is $\ln S_t \sim \mathcal{N}\left(\ln S_0 + (\mu - \frac{1}{2}\sigma^2)t, \sigma\sqrt{t}\right)$ and the dynamics of the underlying can be written as

$$S_t = S_0 \exp\left((\mu - \frac{1}{2}\sigma^2)t + \sigma W_t\right)\ . \tag{3.4$'$}$$

The Black-Scholes option price of European type at time t is then a function of the underlying S_t, strike K, maturity $\tau = T - t$, continuously compounded deterministic interest rate r and constant volatility σ:

$$V_t^{BS} = \eta[S_t\Phi(\eta d_+) - \exp(-r\tau)K\Phi(\eta d_-)]\ , \tag{3.5}$$

with

$$\eta = \begin{cases} +1 & \text{for a call,} \\ -1 & \text{for a put} \end{cases}$$

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} \exp\left(-\frac{u^2}{2}\right) du$$

$$d_+ = \frac{\ln \frac{S_t}{K} + (r + \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}} \qquad d_- = \frac{\ln \frac{S_t}{K} + (r - \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}} .$$

The main result (3.5) is obtained by constructing a riskless arbitrage-free portfolio. At first we derive the Black-Scholes differential equation.

3.3.3 The Black-Scholes Differential Equation

We choose the bank account as the numeraire, that is $\tilde{B}_t \equiv 1$. We are to find the appropriate shift, giving us an equivalent measure $\mathbf{Q}$, under which the discounted price processes are martingales[¶]

$$\tilde{V}_t := \frac{V_t}{B_t} = \mathbf{E}^{\mathbf{Q}}\big[\tilde{V}_T|\mathcal{F}_t\big] . \tag{3.6}$$

Define a process

$$Z_T = \exp\left(-\lambda W_T - \frac{1}{2}\langle \lambda W_T, \lambda W_T\rangle\right) = \exp\left(-\lambda W_T - \frac{1}{2}\lambda^2 T\right) , \tag{3.7}$$

where $\lambda := \frac{\mu - r}{\sigma}$ is called the *market price of volatility risk.*
Since $Law(W_T - W_t) = Law(X)$, for $X \sim \mathcal{N}(0, T-t)$ it follows for the characteristic function of $(W_T - W_t)$ with $u \in \mathbb{R}$

$$\mathbf{E}\big[\exp\big(iu(W_T - W_t)\big)\big] = \mathbf{E}\big[\exp\big(iuX\big)\big] = \exp\left(-\frac{u^2(T-t)}{2}\right) . \tag{3.8}$$

We set $t = 0$ for convenience and observe for (3.7)

$$\mathbf{E}\big[Z_T\big] = \mathbf{E}\left[\exp\big(-\lambda W_T - \frac{1}{2}\lambda^2 T\big)\right] = \exp\big(-\frac{1}{2}\lambda^2 T\big)\mathbf{E}\big[\exp\big(i^2\lambda W_T\big)\big] \tag{3.9}$$

$$= \exp\left(-\frac{1}{2}\lambda^2 T\right)\exp\left(\frac{1}{2}\lambda^2 T\right) = 1 . \tag{3.10}$$

[¶]This property is called the *Risk Neutral Valuation*, see [1], Prop. 6.9.

Thus we are able to apply Girsanov's Theorem and define an equivalent martingale measure $\mathbf{Q}$ by

$$\mathbf{Q}(F) = \mathbf{E}^{\mathbf{P}}\big[Z_T \mathbf{1}_F\big], \quad \forall F \in \mathcal{F}_T \,. \tag{3.11}$$

It also follows from Girsanov's theorem that $\hat{W}_t := W_t + \lambda t$ is a $\mathbf{Q}$-martingale. And (3.4′) writes in terms of $\hat{W}_t$ as

$$S_T = S_t \exp\left((r - \frac{1}{2}\sigma^2)\tau + \sigma(\hat{W}_T - \hat{W}_t) \right) ,$$

or in terms of a stochastic variable $Z \sim \mathcal{N}(0,1)$ with density $\phi(z) = \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-z^2}{2}\right)$

$$S_T = S_t \exp\left((r - \frac{1}{2}\sigma^2)\tau + \sigma\sqrt{\tau} Z \right) . \tag{3.12}$$

Applying the Feynman-Kač Stochastic Representation Formula and (3.12) we receive for the case of a European call option

$$V_t = \exp(-r\tau)\mathbf{E}^{\mathbf{Q}}\big[h(S_T)|\mathcal{F}_t\big] \,. \tag{3.13}$$

In the following we assume S_t to be Markovian.‖ Let us denote with $V = v(t, S_t)$ the value of the payoff of a contingent claim $h := \eta(S_T - K)^+$. By Itô's Lemma

$$V_{\hat{t}} - V_0 = \int_0^{\hat{t}} \left(\mu S_u \, \partial_s v + \partial_t v + \frac{1}{2}\sigma^2 S_u^2 \, \partial_{ss} v \right) du + \int_0^{\hat{t}} \sigma S_u \, \partial_s v \, dW_u \,, \tag{3.14}$$

where ∂ denotes the corresponding partial derivative.
Furthermore, the stochastic part of the change in the option price is assumed to be perfectly correlated with the underlying changes. This allows to set up a portfolio Π consisting of a short position in a claim and a long position of Δ units of the underlying:

$$\Pi = -V + \Delta S \,. \tag{3.15}$$

‖The future behavior of the process S_t given what has happened up to time t, is the same as the behavior obtained when starting the process at S_t; see [20] p. 109.

A change in the value of the portfolio over a time interval $\hat{t}$

$$\Pi_{\hat{t}} - \Pi_0 = -(V_{\hat{t}} - V_0) + \int_0^{\hat{t}} \Delta_u dS_u \,. \tag{3.16}$$

Substituting (3.4) and (3.14) into (3.16) and rearranging

$$\Pi_{\hat{t}} - \Pi_0 = \int_0^{\hat{t}} \Big(-\mu S_u \,\partial_s v - \partial_t v - \frac{1}{2}\sigma^2 S_u^2 \,\partial_{ss} v + \Delta_u \mu S_u \Big) du + \int_0^{\hat{t}} \Big(-\sigma S_u \,\partial_s v + \Delta_u \sigma S_u \Big) dW_u \,. \tag{3.17}$$

Choosing $\Delta = \partial_s v$ (3.17) becomes

$$\Pi_{\hat{t}} - \Pi_0 = \int_0^{\hat{t}} \Big(-\partial_t v - \frac{1}{2}\sigma^2 S_u^2 \,\partial_{ss} v \Big) du \,. \tag{3.18}$$

To exclude any arbitrage opportunities the portfolio Π must earn at a risk-free rate r[**]

$$\Pi_{\hat{t}} - \Pi_0 = \int_0^{\hat{t}} r \Pi_u du \,. \tag{3.19}$$

Setting (3.18) equal to (3.19) and substituting from (3.15) we obtain the Black-Scholes differential equation

$$\partial_t v + rs \,\partial_s v + \frac{1}{2}\sigma^2 s^2 \,\partial_{ss} v - rv = 0 \,, \tag{3.20}$$

where v and its derivatives are evaluated at (t, S_t). The boundary condition for the partial differential equation above is given by

$$v(T, S_T) = h(S_T) \,.$$

[**]This follows from simple arguments excluding arbitrage.

The solution to (3.20) can now be derived with (3.13).

$$
\begin{aligned}
V_t &= \exp(-r\tau)\mathbf{E}^{\mathbf{Q}}\big[h(S_T)|\mathcal{F}_t\big] = \exp(-r\tau)\mathbf{E}^{\mathbf{Q}}\big[h(S_T)|S_t = s\big] \\
&= \exp(-r\tau)\int_{-\infty}^{+\infty}\left[s\exp\left((r-\frac{1}{2}\sigma^2)\tau + \sigma\sqrt{\tau}z\right) - K\right]\mathbf{1}_{\{z_0>K\}}\phi(z)dz
\end{aligned}
$$

with $\hat{r} := (r - \frac{1}{2}\sigma^2)$ and $z_0 := (\ln\frac{K}{s} - \hat{r}\tau)/(\sigma\sqrt{\tau})$ we get

$$
\begin{aligned}
&= \exp(-r\tau)\left(\int_{z_0}^{+\infty} s\exp\left(\hat{r}\tau + \sigma\sqrt{\tau}z\right)\phi(z)dz - K\int_{z_0}^{+\infty}\phi(z)dz\right) \\
&= \exp(-r\tau)\left(\frac{s\exp(\hat{r}\tau)}{\sqrt{2\pi}}\int_{z_0}^{+\infty}\exp\left(-\frac{1}{2}(z-\sigma\sqrt{\tau})^2 + \frac{1}{2}\sigma^2\tau\right)dz - K\Phi(-z_0)\right) \\
&= \exp(-r\tau)\left(s\exp(r\tau)\int_{z_0}^{+\infty}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}(z-\sigma\sqrt{\tau})^2\right)dz - K\Phi(-z_0)\right) \\
&= s\int_{z_0}^{+\infty}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}(z-\sigma\sqrt{\tau})^2\right)dz - \exp(-r\tau)K\Phi(-z_0)
\end{aligned}
$$

and recognizing the integrand as the density function of $Z' \sim \mathcal{N}(\sigma\sqrt{\tau}, 1)$ the result (3.5) for a call option follows with $Z' - \sigma\sqrt{\tau} \sim \mathcal{N}(0,1)$. Put premium follows then with the put-call parity.

3.3.4 Implied Volatility

The Black-Scholes model is often chosen as a starting point. However, empirical results have revealed that the model experiences heavy deviations from the realities of current options markets – the crucial Black-Scholes assumption of constant volatility misprizes a number of options systematically. There are several concepts of volatility to fix this problem. Two of them are briefly introduced below.

Historical volatility is based on historical market data over some time period in the past. It can be computed as the standard deviation of the natural logarithm of close-to-close prices of the underlying:[††]

$$
\vartheta := \frac{1}{n-1}\sum_{i=1}^{n}\left(\log\left(\frac{x_i}{x_{i-1}}\right)\right)^2 - \frac{1}{n(n-1)}\left(\sum_{i=1}^{n}\log\left(\frac{x_i}{x_{i-1}}\right)\right)^2,
$$

[††]See [9], pp. 239-240.

where $x_1, \ldots, x_n$ are the close-to-close prices, equally spaced with distance Δt, which is measured in years. The denominator $n-1$ is chosen to form an unbiased estimator and for an estimator for the historical volatility follows

$$\hat{\sigma}^h := \sqrt{\frac{\vartheta}{\Delta t}} \, .$$

A problem coming up is the appropriate period of time over which the estimation should be calculated: a very large set could include many old data, which are of little importance for the future volatility, since volatility changes over time.

A direct measurement of volatility is thus difficult in practice. Since we assume the market is efficient, it provides us with proper option premiums. It is also aware of the proper volatility. This feature forms the concept of implied volatility.

Definition 6 (Implied Volatility)
Implied volatility I is the volatility, for which the Black-Scholes price equals the market price

$$V^{BS}(t, S_t; K, T; I) = V^{MK}. \tag{3.21}$$

Note, that the put-call parity implies that puts and calls with the same strike have identical implied volatilities. Implied volatility can be thought of as a consensus among the market participants about the future level of volatility – assuming a fair allocation of information, as well as a same model used by all market participants for pricing options.

A concept closely related to implied volatility is *smile effect* – volatility obtained from market prices is often U-shaped, having its minimum *near-the-money*, often defined as an interval, for which

$$0.95 \leq \mathfrak{m} \leq 1.05 \, .$$

Deviation of implied volatility from a constant Black-Scholes volatility can be viewed as the risk premium payable to the holder of the short position, which indirectly implies volatility to be fungible.[‡‡] Trading volatility is accomplished for example by *selling vega* – a position achieved by selling an option. This technique makes profit if the underlying exhibits no movements or falls. Trading *a time spread* – is a portfolio, consisting of long and short options with different expiries and, typically, same exercise price. Long time spreads – buying a long-dated option and selling a short-dated one – become more worthy with increasing volatility, since a long-dated option has a larger vega.

[‡‡]A number of products allow brokers to trade pure volatility, for example volatility or variance swaps, volatility indexes or futures on volatility indexes.

As indicated by several researchers, volatility tends to be mean-reverting (e.g. see a current research on implied volatility indices [5], [9], p. 377 or [13], p. 292). A unique implied volatility given the Black-Scholes price can be found with numerical procedures (such as Newton-Raphson used by Matlab), since

$$\frac{\partial C^{BS}}{\partial \sigma} = \Lambda > 0 \, .$$

This legitimates a market standard to quote prices in terms of implied volatilities.
Most of the time implied volatility is larger than historical. Implied volatility increases in time to maturity and becomes less profound – compare Figure B.1.

3.3.5 The Greeks

Traders are interested in risks connected to a particular option. The sensitivities of an option can be described by partial derivatives of the option premium with respect to the model and the parameters.
We list the most commonly used of them for vanilla options in the Table 3.1, where $\Phi(z)$, η, d_+ and d_- are defined as in (3.5) and

$$\phi(z) = \frac{\partial \Phi(z)}{\partial z} = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) \, .$$

Γ and Ξ give the curvature of Δ and Λ correspondingly. The Greeks containing partial derivatives with respect to volatility, measure sensitivities to misspecifications within the model. Other Greeks, as $\Theta = \partial V/\partial t$ and $\rho = \partial V/\partial r$, are less important – in the case of Theta we have a deterministic time-decay and the magnitude of Rho is extremely small.

Hedging against any of the sensitivities requires another option and the underlying itself. To eliminate the short-term dependancies on any of the Greeks, hedgers are required to set up an appropriate portfolio of the underlying and other derivatives.

Some useful relations and notations.
For an option with strike K one has

$$\begin{gathered} \Delta(t;K)_{CALL} - \Delta(t;K)_{PUT} = 1 \\ 0 \le \Delta(t;K)_{CALL} \le 1 \, . \end{gathered}$$

Especially Foreign Exchange markets speak about plain vanilla options in terms of Delta

Greek		Representation
Delta	$\Delta = \dfrac{\partial V}{\partial s}$	$\eta \Phi(\eta d_+)$
Gamma	$\Gamma = \dfrac{\partial^2 V}{\partial s^2}$	$\dfrac{1}{S_t \sigma \sqrt{\tau}} \phi(d_+)$
Vega	$\Lambda = \dfrac{\partial V}{\partial \sigma}$	$S_t \sqrt{\tau} \phi(d_+)$
Volga	$\Xi = \dfrac{\partial^2 V}{\partial \sigma^2}$	$\dfrac{S_t \sqrt{\tau} d_+ d_-}{\sigma} \phi(d_+)$
Vanna	$\Psi = \dfrac{\partial^2 V}{\partial s \partial \sigma}$	$-\dfrac{d_-}{\sigma} \phi(d_+)$
Speed	$\Upsilon = \dfrac{\partial^3 V}{\partial s^3}$	$-\left(\dfrac{d_+}{\sigma \sqrt{\tau}} + 1 \right) \dfrac{\phi(d_+)}{S_t^2 \sigma \sqrt{\tau}}$
Dual Delta	$\Delta^* = \dfrac{\partial V}{\partial K}$	$-\eta \exp(-r\tau) \Phi(\eta d_-)$

Table 3.1: Greeks for European options

and quote those in terms of volatility. It abstracts from strike and current underlying price, giving a transparent and a user-friendly method. A $k\Delta$ option, is an option whose Δ is $k/100$ for a call and $-k/100$ for a put. For detailed relationships among the Greeks see [14].

4 Vanna-Volga Method

The Vanna-Volga method is commonly used by market participants trading foreign exchange (FX) options, which arises from the fact, that the FX market has only few active quotes for each maturity:

0Δ *straddle* is a long call and a long put with the same strike and expiration date – trader bets on raising volatility. The premium of a straddle yields information about the expected volatility of the underlying – higher volatility means higher profit, and as a result a higher premium.

Risk-reversal is a long out-of-the-money call and a short out-of-the-money put with a symmetric Δ. Most common risk-reversals use 25Δ options. Traders see a positive risk-reversal as an indicator of a bullish market, since calls are more expensive than puts, and vice-versa.

Vega-weighted butterfly is constructed by a short at-the-money straddle and a long 25Δ strangle.* A buyer of a vega-weighted butterfly profits under a stable underlying. A straddle together with a strangle give simple techniques to trade volatility.

The AtM volatility σ_{AtM} is then derived as the volatility of the 0Δ straddle and the volatilities of the risk-reversal (RR) and the vega-weighted butterfly (VWB) are subject to following relations:†

$$\begin{aligned} \sigma_{RR} &= \sigma_{25\Delta CALL} - \sigma_{25\Delta PUT} \\ \sigma_{VWB} &= \frac{1}{2}(\sigma_{25\Delta CALL} + \sigma_{25\Delta PUT}) - \sigma_{AtM} \, . \end{aligned}$$

Implied volatility of a risk-reversal incorporates information on the skew of the implied volatility curve, whereas that of a strangle – on the kurtosis.

* *Strangle* is set up by a long $k\Delta$ call and a long $k\Delta$ put. The strategy is less expensive than a straddle, being profitable for a higher volatility.

† See [21], p. 35.

With the volatilities received in that way, Vanna-Volga allows us to reconstruct the whole smile for a given maturity. At first one evaluates data received with this method as proposed by Castagna and Mercurio in [3]. In the research the authors applied Vanna-Volga on EUR/USD exchange rate and obtained good results for strikes with moneyness $0.9 < \mathfrak{m} < 1.1$.

We evaluated the method for call options on EURO STOXX 50 for a time period of one month with two different maturities. In this chapter we only introduce the results obtained within Vanna-Volga method. An interpretation and further discussion of (4.4) and (4.5) are given in Chapter 5.2 and Chapter 5.3 correspondingly.

4.1 Option Premium

As already mentioned, moneyness is defined as $\mathfrak{m} := \frac{K_j}{S_t}$. A daily set of strikes, *range of moneyness*, satisfying this requirement $\mathcal{K}_t := \{K_j \mid 0.8 < \mathfrak{m} < 1.2, \mathcal{F}_t\}$, with $K_i < K_j$, for $i < j$ is totally ordered. The Black-Scholes price of a European call option at time t, with maturity T and strike K is denoted by $C^{BS}(t;K)$; the corresponding settlement price by $C^{MK}(t;K)$.

We choose some option, *the reference option*, and use its implied volatility for calculation of the Black-Scholes prices, for Black-Scholes assumes constant volatility. By $\overline{\sigma}$ we denote the implied volatility of the reference option, the *reference volatility*. At first we compute the theoretical values for Vega, Volga and Vanna for $\mathcal{K}_t$ using the formulas from Table 3.1.

Our aim is to construct a weighted portfolio consisting of three liquid options with strikes K_1, K_2, K_3. Since those options are frequently traded, their implied volatilities σ_1, σ_2 and σ_3 are precise and can be calculated easily. The constructed portfolio should be vega, volga and vanna neutral with respect to an illiquid option with strike K. The time weights $x_1(t;K)$, $x_2(t;K)$, $x_3(t;K)$ then make the portfolio instantaneously hedged up to the second order derivatives.

$$\begin{aligned}\Lambda(t;K) &= \sum_{i=1}^{3} x_i(t;K_i)\cdot\Lambda(t;K_i)\\ \Xi(t;K) &= \sum_{i=1}^{3} x_i(t;K_i)\cdot\Xi(t;K_i)\\ \Psi(t;K) &= \sum_{i=1}^{3} x_i(t;K_i)\cdot\Psi(t;K_i)\end{aligned}\tag{4.1}$$

or in matrix notation

$$v = A\cdot x\,.\tag{4.1$'$}$$

Proposition 1

The system (4.1′) admits a unique solution $x = A^{-1}\cdot v$, with x_i given by

$$\begin{aligned}x_1(t;K) &= \frac{\Lambda(t;K)}{\Lambda(t;K_1)}\frac{\ln\frac{K_2}{K}\ln\frac{K_3}{K}}{\ln\frac{K_2}{K_1}\ln\frac{K_3}{K_1}}\\ x_2(t;K) &= \frac{\Lambda(t;K)}{\Lambda(t;K_2)}\frac{\ln\frac{K}{K_1}\ln\frac{K_3}{K}}{\ln\frac{K_2}{K_1}\ln\frac{K_3}{K_2}}\\ x_3(t;K) &= \frac{\Lambda(t;K)}{\Lambda(t;K_3)}\frac{\ln\frac{K}{K_1}\ln\frac{K}{K_2}}{\ln\frac{K_3}{K_1}\ln\frac{K_3}{K_2}}\end{aligned}\tag{4.2}$$

Proof

$$\begin{aligned}|A| = &-\frac{\Lambda(t;K_1)\Lambda(t;K_2)\Lambda(t;K_3)}{S_t\sigma^2\sqrt{\tau}}\cdot\big[d_-(K_3)d_+(K_2)d_-(K_2)+d_-(K_1)d_+(K_3)d_-(K_3)\\ &-d_+(K_1)d_-(K_1)d_-(K_3)-d_+(K_3)d_-(K_3)d_-(K_2)-d_-(K_1)d_+(K_2)d_-(K_2)\\ &+d_+(K_1)d_-(K_1)d_-(K_2)\big]\\ = &-\frac{\Lambda(t;K_1)\Lambda(t;K_2)\Lambda(t;K_3)}{S_t\sigma^5\tau^2}\ln\frac{K_2}{K_1}\ln\frac{K_3}{K_1}\ln\frac{K_3}{K_2}\,.\end{aligned}\tag{4.3}$$

For positive $K_1 < K_2 < K_3$, $|A| < 0$ and the unique solution for (4.1′) follows from Cramer's rule. ■

The option price with an illiquid strike K is then given by

$$\hat{C}(t;K) = C^{BS}(t;K) + \sum_{i=1}^{3} x_i(t;K) \cdot [C^{MK}(t;K_i) - C^{BS}(t;K_i)] \ . \tag{4.4}$$

4.2 Implied Volatility

The implied volatility $\hat{\sigma}_{t;K}$, corresponding to the pricing formula (4.4) is approximated by the sum of the reference volatility $\overline{\sigma}$ and a term including the basic volatilities σ_1, σ_2 and σ_3

$$\hat{\sigma}_{t;K} \approx \overline{\sigma} + \frac{-\overline{\sigma} + \sqrt{\overline{\sigma}^2 + d_+(K)d_-(K)(2\overline{\sigma}D_+(K) + D_-(K))}}{d_+(K)d_-(K)} \tag{4.5}$$

where $d_+(K)$ and $d_-(K)$ are as in (3.5) and

$$\begin{aligned}
D_+(K) &:= \frac{\ln\frac{K_2}{K}\ln\frac{K_3}{K}}{\ln\frac{K_2}{K_1}\ln\frac{K_3}{K_1}}\sigma_1 + \frac{\ln\frac{K}{K_1}\ln\frac{K_3}{K}}{\ln\frac{K_2}{K_1}\ln\frac{K_3}{K_2}}\sigma_2 + \frac{\ln\frac{K}{K_1}\ln\frac{K}{K_2}}{\ln\frac{K_3}{K_1}\ln\frac{K_3}{K_2}}\sigma_3 - \overline{\sigma}\ , \\
D_-(K) &:= \frac{\ln\frac{K_2}{K}\ln\frac{K_3}{K}}{\ln\frac{K_2}{K_1}\ln\frac{K_3}{K_1}}d_+(K_1)d_-(K_1)(\sigma_1 - \overline{\sigma})^2 \\
&\quad + \frac{\ln\frac{K}{K_1}\ln\frac{K_3}{K}}{\ln\frac{K_2}{K_1}\ln\frac{K_3}{K_2}}d_+(K_2)d_-(K_2)(\sigma_2 - \overline{\sigma})^2 + \frac{\ln\frac{K}{K_1}\ln\frac{K}{K_2}}{\ln\frac{K_3}{K_1}\ln\frac{K_3}{K_2}}d_+(K_3)d_-(K_3)(\sigma_3 - \overline{\sigma})^2.
\end{aligned}$$

As pointed out by Castagna and Mercurio in [3], the above approximation for EUR/USD exchange rate options is extremely accurate for $0.9 < \mathfrak{m} < 1.1$, although been asymptotically constant at extreme strikes. Another drawback is that it cannot be defined without the square root. However the radicand is positive in most applications.

4.3 Vanna-Volga Result for EURO STOXX 50

We calculated the resulting option price, together with the implied volatility approximation for all $\binom{|\mathcal{K}_t|}{3}$ combinations of strikes. The best-fitting curve for the implied volatility and the corresponding option premiums are depicted in Figure 4.1 and Figure 4.2. As we see, the approximation delivers very good results for around at-the-money options; Asymptotically constant volatility for deep in-the-money options is obvious. An evaluation of Vanna-Volga and its comparison to the extended method are given in Chapter 6.

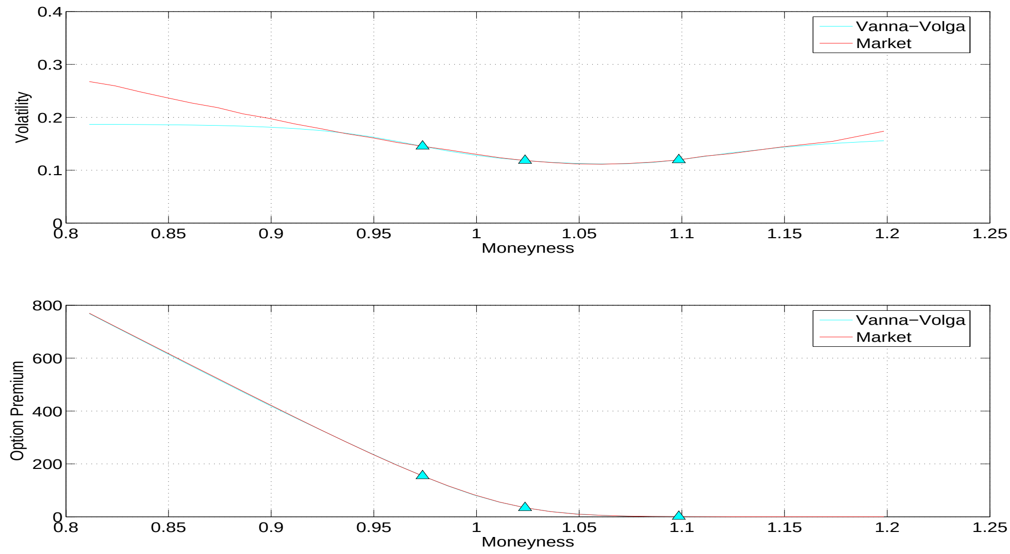

Figure 4.1: The Vanna-Volga method for OESX-1206, $\tau = 0.1205$.
The upper graph shows the volatility approximation (the light blue line) compared to the market implied volatility (red line), where the markers give the positions of the anker points.
The lower, the corresponding option premiums.

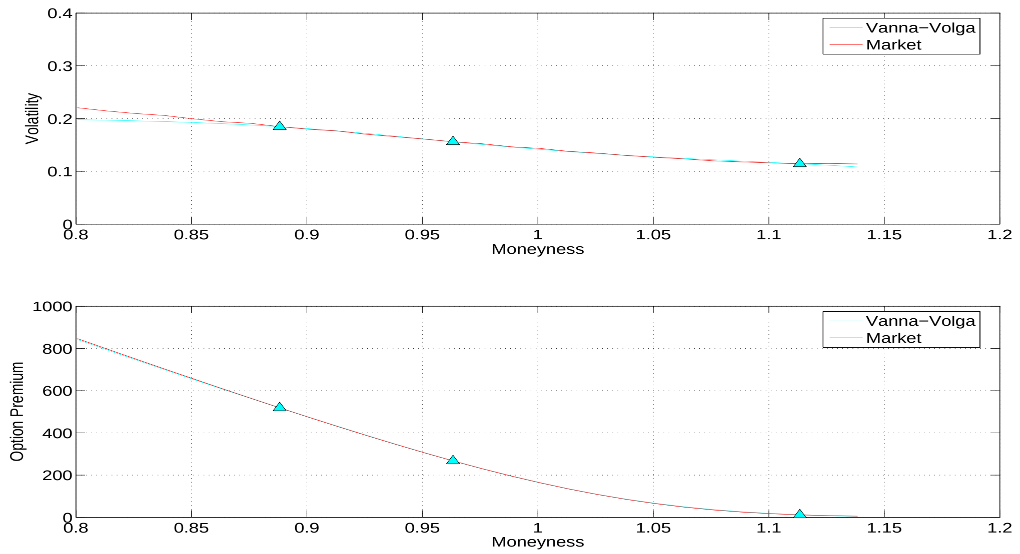

Figure 4.2: The Vanna-Volga method for OESX-0307, $\tau = 0.3999$.
The upper graph shows the volatility approximation (the light blue line) compared to the market implied volatility (red line), where the markers give the positions of the anker points.
The lower, the corresponding option premiums.

5 Investigating Δ-neutrality

As we have observed, the Vanna-Volga method also delivers good results for index options. Asymptotically constant volatility for extreme strikes could come from the lack in Δ-neutrality. A hedge against movements in the underlying is the most "natural" practice among traders. Thus we will investigate consequences of extending the Vanna-Volga method by introducing Δ-neutrality.

5.1 Portfolio Construction

Let us now construct a portfolio of four liquid options with strikes $K_1,\ K_2,\ K_3,\ K_4$. We can easily derive the corresponding implied volatilities $\sigma_i,\ i = \{1 \dots 4\}$. With $C^{MK}(t; K_i),\ K_i \in \mathcal{K}$ we denote option premiums of liquid options. As in Vanna-Volga method we are to find time-dependent weights $x_1(t; K)$, $x_2(t; K)$, $x_3(t; K)$, $x_4(t; K)$, such that the constructed portfolio remains delta, vega, vanna and volga neutral with respect to an illiquid option with strike K

$$\begin{aligned}
\Delta(t; K) &= \sum_{i=1}^{4} x_i(t; K) \cdot \Delta(t; K_i) \\
\Lambda(t; K) &= \sum_{i=1}^{4} x_i(t; K) \cdot \Lambda(t; K_i) \\
\Xi(t; K) &= \sum_{i=1}^{4} x_i(t; K) \cdot \Xi(t; K_i) \\
\Psi(t; K) &= \sum_{i=1}^{4} x_i(t; K) \cdot \Psi(t; K_i)
\end{aligned} \tag{5.1}$$

or using matrix notation, with column vectors v and x

$$v = A \cdot x \ . \tag{5.1'}$$

Being delta, vega, volga and vanna-neutral, the portfolio is furthermore also gamma-neutral. This arises from the Vega-Gamma relationship for European plain-vanilla options (see [12])

$$\frac{1}{2}\sigma\Lambda = \frac{\tau}{2}\sigma^2 S^2 \Gamma\,. \tag{5.2}$$

From the Black-Scholes Differential Equation (3.20) we derive, that it is also Θ neutral. Thus this portfolio is hedged against all Greeks up to the second order.

Now we are to find the appropriate interest rate. This is done by minimizing the put-call parity in r under the assumption of arbitrage-free forward valuation (see [1], p. 91).

The arbitrage-free forward price F_t at time t with maturity T is given by

$$F_t = \exp(r\tau)S_t. \tag{5.3}$$

Then the put-call parity written as a function of r is:

$$p(r) = C_t - P_t + \exp(-r\tau)(K - F_t). \tag{5.4}$$

Minimizing $p(r)^2$ in r by OLS-method we retrieve the interest rate consistent with the market prices

$$\min_r \left[\sum_{K \in \mathcal{K}_t} (C_t - P_t + \exp(-r\tau)(K - F_t))^2 \right]. \tag{5.5}$$

The goodness of this method can be proved by comparing the corresponding futures prices* derived from the recovered interest rate according to (5.3). For that see Table A.1.

The Vanna-Volga method used three strikes to calculate the option price: a 0Δ straddle, a 25Δrisk-reversal and a vega-weighted butterfly. These particular options were chosen, because the FX market has very few active quotes.

This is not the case for index options – being frequently traded, the market becomes even more liquid through market makers.

One important question to address is the appropriate choice of the four strikes. Let $k_t := |\mathcal{K}_t|$ be the number daily strikes that match the moneyness condition (in our case k was about 30). Thus we have to test all $\binom{k_t}{4}$ combinations of strikes. Do quotes within the strike price windows of PML and PMM-intervals deliver better results?

*Under deterministic interest rates futures and forward prices coincide, see [1], p. 92.

5.2 The Pricing Formula

Proposition 2
The system (5.1) admits always a unique solution.

Proof
At first consider that

$$\frac{\Lambda}{\Delta} = \frac{\partial \ln \Delta}{\partial s} S^2 \tau \sigma \ .$$

Since Δ is strictly increasing in S, $\ln \Delta$ is also strictly increasing in S and $\frac{\partial \ln \Delta}{\partial s}$ strictly decreasing. From the duality of S and K follows

$$\Big(S \uparrow \Leftrightarrow K \downarrow \Big) \quad \Longrightarrow \quad \Big(K \downarrow \Leftrightarrow \frac{\partial \ln \Delta}{\partial s} \downarrow \Big) . \tag{5.6}$$

Applying Proposition 1 we write the determinant of A as

$$\begin{aligned}
|A| = & -\Bigg(\Delta_1 \frac{\Lambda_2\Lambda_3\Lambda_4}{S\sigma^5\tau^2} \ln\frac{K_3}{K_2} \ln\frac{K_4}{K_2} \ln\frac{K_4}{K_3} - \Delta_2 \frac{\Lambda_1\Lambda_3\Lambda_4}{S\sigma^5\tau^2} \ln\frac{K_3}{K_1} \ln\frac{K_4}{K_1} \ln\frac{K_4}{K_3} \\
& + \Delta_3 \frac{\Lambda_1\Lambda_2\Lambda_4}{S\sigma^5\tau^2} \ln\frac{K_2}{K_1} \ln\frac{K_4}{K_1} \ln\frac{K_4}{K_2} - \Delta_4 \frac{\Lambda_1\Lambda_2\Lambda_3}{S\sigma^5\tau^2} \ln\frac{K_2}{K_1} \ln\frac{K_3}{K_1} \ln\frac{K_3}{K_2} \Bigg) \\
= & -\frac{\Lambda_1\Lambda_2\Lambda_3\Lambda_4}{S\sigma^5\tau^2} \Bigg(\frac{\Delta_1}{\Lambda_1} \ln\frac{K_3}{K_2} \ln\frac{K_4}{K_2} \ln\frac{K_4}{K_3} - \frac{\Delta_2}{\Lambda_2} \ln\frac{K_3}{K_1} \ln\frac{K_4}{K_1} \ln\frac{K_4}{K_3} \\
& + \frac{\Delta_3}{\Lambda_3} \ln\frac{K_2}{K_1} \ln\frac{K_4}{K_1} \ln\frac{K_4}{K_2} - \frac{\Delta_4}{\Lambda_4} \ln\frac{K_2}{K_1} \ln\frac{K_3}{K_1} \ln\frac{K_3}{K_2} \Bigg)
\end{aligned} \tag{5.7}$$

$$\begin{aligned}
= & -\frac{\Lambda_1\Lambda_2\Lambda_3\Lambda_4}{S^3\sigma^6\tau^3} \Bigg(\frac{\partial s}{\partial \ln \Delta_1} \ln\frac{K_3}{K_2} \ln\frac{K_4}{K_2} \ln\frac{K_4}{K_3} - \frac{\partial s}{\partial \ln \Delta_2} \ln\frac{K_3}{K_1} \ln\frac{K_4}{K_1} \ln\frac{K_4}{K_3} \\
& + \frac{\partial s}{\partial \ln \Delta_3} \ln\frac{K_2}{K_1} \ln\frac{K_4}{K_1} \ln\frac{K_4}{K_2} - \frac{\partial s}{\partial \ln \Delta_4} \ln\frac{K_2}{K_1} \ln\frac{K_3}{K_1} \ln\frac{K_3}{K_2} \Bigg) \ ,
\end{aligned} \tag{5.7$'$}$$

where $\Delta_i := \Delta(S_t, t; K_i)$ and $\Lambda_i := \Lambda(S_t, t; K_i)$.
Simple algebra shows

$$\ln\frac{K_3}{K_2} \ln\frac{K_4}{K_2} \ln\frac{K_4}{K_3} - \ln\frac{K_3}{K_1} \ln\frac{K_4}{K_1} \ln\frac{K_4}{K_3} + \ln\frac{K_2}{K_1} \ln\frac{K_4}{K_1} \ln\frac{K_4}{K_2} - \ln\frac{K_2}{K_1} \ln\frac{K_3}{K_1} \ln\frac{K_3}{K_2} = 0 \ .$$

Summing up, the term before parenthesis in (5.7$'$) is negative and for the coefficients before logarithm terms in parenthesis we observe with (5.6)

$$\frac{\partial s}{\partial \ln \Delta_i} > \frac{\partial s}{\partial \ln \Delta_j} \quad \text{, for } i < j$$

since $K_i < K_j$, for $i < j$.
Thus, $|A| < 0$.
The unique solution for (5.1) follows from Cramer's Rule with (5.7)

$$\begin{aligned}
x_1(t;K) &= \frac{\Lambda_K}{\Lambda_1} \frac{\ln\frac{K_4}{K_3}\left(\frac{\Delta_K}{\Lambda_K}\ln\frac{K_3}{K_2}\ln\frac{K_4}{K_2} - \frac{\Delta_2}{\Lambda_2}\ln\frac{K_3}{K}\ln\frac{K_4}{K}\right) + \ln\frac{K_2}{K}\left(\frac{\Delta_3}{\Lambda_3}\ln\frac{K_4}{K}\ln\frac{K_4}{K_2} - \frac{\Delta_4}{\Lambda_4}\ln\frac{K_3}{K}\ln\frac{K_3}{K_2}\right)}{\ln\frac{K_4}{K_3}\left(\frac{\Delta_1}{\Lambda_1}\ln\frac{K_3}{K_2}\ln\frac{K_4}{K_2} - \frac{\Delta_2}{\Lambda_2}\ln\frac{K_3}{K_1}\ln\frac{K_4}{K_1}\right) + \ln\frac{K_2}{K_1}\left(\frac{\Delta_3}{\Lambda_3}\ln\frac{K_4}{K_1}\ln\frac{K_4}{K_2} - \frac{\Delta_4}{\Lambda_4}\ln\frac{K_3}{K_1}\ln\frac{K_3}{K_2}\right)} \\
x_2(t;K) &= \frac{\Lambda_K}{\Lambda_2} \frac{\ln\frac{K_4}{K_3}\left(\frac{\Delta_1}{\Lambda_1}\ln\frac{K_3}{K}\ln\frac{K_4}{K} - \frac{\Delta_K}{\Lambda_K}\ln\frac{K_3}{K_1}\ln\frac{K_4}{K_1}\right) + \ln\frac{K}{K_1}\left(\frac{\Delta_3}{\Lambda_3}\ln\frac{K_4}{K_1}\ln\frac{K_4}{K} - \frac{\Delta_4}{\Lambda_4}\ln\frac{K_3}{K_1}\ln\frac{K_3}{K}\right)}{\ln\frac{K_4}{K_3}\left(\frac{\Delta_1}{\Lambda_1}\ln\frac{K_3}{K_2}\ln\frac{K_4}{K_2} - \frac{\Delta_2}{\Lambda_2}\ln\frac{K_3}{K_1}\ln\frac{K_4}{K_1}\right) + \ln\frac{K_2}{K_1}\left(\frac{\Delta_3}{\Lambda_3}\ln\frac{K_4}{K_1}\ln\frac{K_4}{K_2} - \frac{\Delta_4}{\Lambda_4}\ln\frac{K_3}{K_1}\ln\frac{K_3}{K_2}\right)} \\
x_3(t;K) &= \frac{\Lambda_K}{\Lambda_3} \frac{\ln\frac{K_4}{K}\left(\frac{\Delta_1}{\Lambda_1}\ln\frac{K}{K_2}\ln\frac{K_4}{K_2} - \frac{\Delta_2}{\Lambda_2}\ln\frac{K}{K_1}\ln\frac{K_4}{K_1}\right) + \ln\frac{K_2}{K_1}\left(\frac{\Delta_K}{\Lambda_K}\ln\frac{K_4}{K_1}\ln\frac{K_4}{K_2} - \frac{\Delta_4}{\Lambda_4}\ln\frac{K}{K_1}\ln\frac{K}{K_2}\right)}{\ln\frac{K_4}{K_3}\left(\frac{\Delta_1}{\Lambda_1}\ln\frac{K_3}{K_2}\ln\frac{K_4}{K_2} - \frac{\Delta_2}{\Lambda_2}\ln\frac{K_3}{K_1}\ln\frac{K_4}{K_1}\right) + \ln\frac{K_2}{K_1}\left(\frac{\Delta_3}{\Lambda_3}\ln\frac{K_4}{K_1}\ln\frac{K_4}{K_2} - \frac{\Delta_4}{\Lambda_4}\ln\frac{K_3}{K_1}\ln\frac{K_3}{K_2}\right)} \\
x_4(t;K) &= \frac{\Lambda_K}{\Lambda_4} \frac{\ln\frac{K}{K_3}\left(\frac{\Delta_1}{\Lambda_1}\ln\frac{K_3}{K_2}\ln\frac{K}{K_2} - \frac{\Delta_2}{\Lambda_2}\ln\frac{K_3}{K_1}\ln\frac{K}{K_1}\right) + \ln\frac{K_2}{K_1}\left(\frac{\Delta_3}{\Lambda_3}\ln\frac{K}{K_1}\ln\frac{K}{K_2} - \frac{\Delta_K}{\Lambda_K}\ln\frac{K_3}{K_1}\ln\frac{K_3}{K_2}\right)}{\ln\frac{K_4}{K_3}\left(\frac{\Delta_1}{\Lambda_1}\ln\frac{K_3}{K_2}\ln\frac{K_4}{K_2} - \frac{\Delta_2}{\Lambda_2}\ln\frac{K_3}{K_1}\ln\frac{K_4}{K_1}\right) + \ln\frac{K_2}{K_1}\left(\frac{\Delta_3}{\Lambda_3}\ln\frac{K_4}{K_1}\ln\frac{K_4}{K_2} - \frac{\Delta_4}{\Lambda_4}\ln\frac{K_3}{K_1}\ln\frac{K_3}{K_2}\right)}
\end{aligned} \tag{5.8}$$

where $\Delta_i := \Delta(S_t, t; K_i)$, $\Delta_K := \Delta(S_t, t; K)$ and $\Lambda_i := \Lambda(S_t, t; K_i)$, $\Lambda_K := \Lambda(S_t, t; K)$. ■

Then the option premium $\hat{C}(t;K)$ of the illiquid option with strike K is:

$$\hat{C}(t;K) = C^{BS}(t;K) + \sum_{i=1}^{4} x_i(t;K) \cdot [C^{MK}(t;K_i) - C^{BS}(t;K_i)] \tag{5.9}$$

or substituting from (5.1′) and y a column vector, with $y_i := C^{MK}(t;K_i) - C^{BS}(t;K_i)$

$$\hat{C}(t;K) = C^{BS}(t;K) + (A^{-1}v)'\, y = C^{BS}(t;K) + v'w\ , \tag{5.9′}$$

where $w := (A')^{-1}\, y$.

Properties of (5.9):

1. The option premium approximation formula is a inter or extrapolation formula of $\hat{C}(t;K)$. Thus on the one hand we are able to price far out-of-the-money, as well

as deep in-the-money options. On the other hand we retrieve premiums even for options that are not offered by the market place.

2. The four anker points $C^{MK}(t;K_i),\ i = \{1\dots4\}$ are matched exactly, since for $K = K_j$ we have (compare Table A.2)

$$x_i(t;K) = \begin{cases} 1 & \text{for } i = j, \\ 0 & \text{otherwise} \end{cases}$$

3. However, the pricing formula delivers not always arbitrage-free prices, that is

$$\hat{C}(t;K_i) < \hat{C}(t;K_j), \quad \text{for some } i < j,\ K_i \in \mathcal{K}\,.$$

4. Following no-arbitrage conditions still hold
   a) $\hat{C}(t;K) \in \mathcal{C}^2((0,+\infty))$
   b) $\lim_{K\to+\infty} \hat{C}(t;K) = 0$

This is an economic interpretation of the pricing formula (5.9′):
The vector w is interpreted as a vector of premiums of the market prices that must be attached to the Greeks in order to adjust the Black-Scholes price of liquid options. This adjustment is called an *over-hedge*. With this interpretation, Δ, Λ, Ξ and Ψ can be seen as proxies for certain risks – volga correction for the kurtosis and vanna correction for the skew. Traders, willing to offload these risks to another party, should compensate them; those bringing the risks into the market, should pay for them. The market cost of such a protection form the weighted excess one has to add to the theoretical Black-Scholes price.

5.3 Derivation of the Implied Volatility

To emphasize the dependance on the volatility we rewrite (5.9) as

$$\hat{C}(t;K) = C^{BS}(t;K;\overline{\sigma}) + \sum_{i=1}^{4} x_i(t;K)\cdot\left[C^{MK}(t;K_i;\sigma_i) - C^{BS}(t;K_i;\overline{\sigma})\right]. \tag{5.10}$$

We approximate the option premium $\hat{C}(t;K)$ by the second order Taylor expansion of (5.10) in $\overline{\sigma}$:

$$
\begin{aligned}
\hat{C}(t;K;\sigma_{1,2,3,4}) \approx C^{BS}(t;K;\overline{\sigma}) + \sum_{i=1}^{4} x_i(t;K) \cdot \Big[\underbrace{C^{MK}(t;K_i;\overline{\sigma}) - C^{BS}(t;K_i;\overline{\sigma})}_{(*)} \Big] \\
+ \frac{\partial C^{BS}(t;K;\overline{\sigma})}{\partial \sigma}(\overline{\sigma} - \overline{\sigma}) \\
+ \sum_{i=1}^{4} x_i(t;K) \cdot \left[\frac{\partial C^{MK}(t;K_i;\overline{\sigma})}{\partial \sigma}(\sigma_i - \overline{\sigma}) - \frac{\partial C^{BS}(t;K_i;\overline{\sigma})}{\partial \sigma}(\overline{\sigma} - \overline{\sigma}) \right] \\
+ \frac{1}{2} \frac{\partial^2 C^{BS}(t;K;\overline{\sigma})}{\partial \sigma^2}(\overline{\sigma} - \overline{\sigma})^2 \\
+ \frac{1}{2} \sum_{i=1}^{4} x_i(t;K) \cdot \left[\frac{\partial^2 C^{MK}(t;K_i;\overline{\sigma})}{\partial \sigma^2}(\sigma_i - \overline{\sigma})^2 - \frac{\partial^2 C^{BS}(t;K_i;\overline{\sigma})}{\partial \sigma^2}(\overline{\sigma} - \overline{\sigma})^2 \right] \\
= C^{BS}(t;K;\overline{\sigma}) + \sum_{i=1}^{4} x_i(t;K) \cdot \frac{\partial C^{MK}(t;K_i;\overline{\sigma})}{\partial \sigma}(\sigma_i - \overline{\sigma}) \\
+ \frac{1}{2} \cdot \sum_{i=1}^{4} x_i(t;K) \cdot \frac{\partial^2 C^{MK}(t;K_i;\overline{\sigma})}{\partial \sigma^2}(\sigma_i - \overline{\sigma})^2 \\
= C^{BS}(t;K;\overline{\sigma}) + \sum_{i=1}^{4} x_i(t;K) \cdot \left[\Lambda(t;K_i;\overline{\sigma})(\sigma_i - \overline{\sigma}) + \frac{1}{2} \cdot \Xi(t;K_i;\overline{\sigma})(\sigma_i - \overline{\sigma})^2 \right] ,
\end{aligned}
\tag{5.11}
$$

with $(*)$ vanishing, since the market price of options under constant volatility $\overline{\sigma}$ equals the Black-Scholes price (compare (7.13)).

On the over hand the market price of the illiquid option is:

$$
\begin{aligned}
\hat{C}(t;K;\hat{\sigma}_{t;K}) \approx C^{MK}(t;K;\overline{\sigma}) + \frac{\partial C^{MK}(t;K;\overline{\sigma})}{\partial \sigma}(\hat{\sigma}_{t;K} - \overline{\sigma}) + \frac{1}{2} \cdot \frac{\partial^2 C^{MK}(t;K;\overline{\sigma})}{\partial \sigma^2}(\hat{\sigma}_{t;K} - \overline{\sigma})^2 \\
= C^{BS}(t;K;\overline{\sigma}) + \Lambda(t;K;\overline{\sigma})(\hat{\sigma}_{t;K} - \overline{\sigma}) + \frac{1}{2} \cdot \Xi(t;K;\overline{\sigma})(\hat{\sigma}_{t;K} - \overline{\sigma})^2 ,
\end{aligned}
\tag{5.12}
$$

where $C^{MK}(t;K;\overline{\sigma})$ turns into $C^{BS}(t;K;\overline{\sigma})$ for the same reason as in $(*)$. The same is true for its derivatives.

The implied volatility $\hat{\sigma}_{t;K}$ follows by equating (5.11) and (5.12) and solving the second-order algebraic equation

$$\hat{\sigma}^{\pm}_{t;K} \approx \overline{\sigma} + \frac{-\Lambda(t;K;\overline{\sigma}) \pm \sqrt{\Lambda(t;K;\overline{\sigma})^2 + 2 \cdot \Xi(t;K;\overline{\sigma}) \cdot \kappa}}{\Xi(t;K;\overline{\sigma})}\,, \tag{5.13}$$

where

$$\kappa := \sum_{i=1}^{4} x_i(t;K) \cdot \left[\Lambda(t;K_i;\overline{\sigma})(\sigma_i - \overline{\sigma}) + \frac{1}{2} \cdot \Xi(t;K_i;\overline{\sigma})(\sigma_i - \overline{\sigma})^2 \right] .$$

Matching the anker volatilities exactly, the formula above gives an easy to implement approximation of the implied volatility. Empirical tests have shown, that the second solution $\hat{\sigma}^{-}_{t;K}$ delivers a flat structure.

The formula above gives us an comfortable way to derive implied volatilities. A noticeable drawback is its dependence on rigorously derived option premiums. As already mentioned, option premiums are not always arbitrage-free. This has severe consequences on the derivation of the implied volatility. However, we are able to provide a control tool for the slope of the implied volatility curve.

We use the definition and properties of implied volatility, which itself is a function of moneyness. For a fixed t and thus constant S_t equation (3.21) can be written in terms of moneyness as

$$C^{MK} = C^{BS}(I(\mathfrak{m}); \mathfrak{m}) .$$

Taking the derivative with respect to $\mathfrak{m}$ and noting that C^{MK} is decreasing in K we obtain

$$\begin{aligned} \frac{\partial C^{MK}}{\partial \mathfrak{m}} = \frac{\partial C^{BS}(I(\mathfrak{m}); \mathfrak{m})}{\partial \sigma} \cdot \frac{\partial I}{\partial \mathfrak{m}} + \frac{\partial C^{BS}(I(\mathfrak{m}); \mathfrak{m})}{\partial \mathfrak{m}} &\leq 0 \\ S\sqrt{\tau}\phi(d_+) \cdot \frac{\partial I}{\partial \mathfrak{m}} - S\exp(-r\tau)\Phi(d_-) &\leq 0 \end{aligned}$$

giving us the upper bound for the slope of the implied volatility curve. Similar derivation for P^{MK}, which is increasing in K, provides us with a lower bound. Altogether we get:

$$-\sqrt{\frac{2\pi}{\tau}} \exp\big(-r\tau + \frac{d_+}{2}\big)\Phi(d_-) \leq \frac{\partial I}{\partial \mathfrak{m}} \leq \sqrt{\frac{2\pi}{\tau}} \exp\big(-r\tau + \frac{d_+}{2}\big)\Phi(d_-) .$$

5.4 Justification

In this section we give a justification for (5.9) using Itô's-formula.

Let us assume, a function $\hat{C}_t$ depends on S_t, $\tau = T - 0$, K and σ_t. We allow σ_t to be not only time-dependand but also possibly stochastic. Applying Itô's Lemma we get for the value of a vanilla option $\hat{C}(S_t, t; K; \sigma_t)$

$$\begin{aligned}\hat{C}(S_T; K; \sigma_t) = \hat{C}(S_0; K; \sigma_0) \quad &+ \quad \int_0^T \frac{\partial C}{\partial s} dS + \int_0^T \frac{\partial C}{\partial t} dt + \int_0^T \frac{1}{2} \frac{\partial^2 C}{\partial s^2} d\langle S, S\rangle \\ &+ \quad \int_0^T \frac{\partial C}{\partial \sigma} d\sigma + \int_0^T \frac{\partial^2 C}{\partial s \partial \sigma} d\langle S, \sigma\rangle + \int_0^T \frac{1}{2} \frac{\partial^2 C}{\partial \sigma} d\langle \sigma, \sigma\rangle \, .\end{aligned}$$

The first three integral terms form the Itô-expansion of the Black-Scholes price, the theoretical price. The latter three come from the stochastic volatility and give an adjustment to the theoretical price. Hence for an arbitrary option $C(S_t; K_i; \sigma_t, \tau)$

$$C^{MK}(S_t; K_i; \sigma_t, \tau) - C^{BS}(S_t; K_i; \sigma_t, \tau)$$

gives its stochastic part, which can be approximated by a delta, vega, vanna and volga neutral portfolio

$$\sum_{i=1}^{4} x_i(t; K) \cdot [C^{MK}(t; K_i) - C^{BS}(t; K_i)] \, .$$

6 Comparison

6.1 Results

Time series from 1.-30. November 2006 were chosen as a basis for the comparison – a time span containing 22 trading days. In order to compare the results of both pricing procedures December 2006 (OESX-0612) and March 2007 (OESX-0307) were selected as option expiries – 44-15 and 135-106 days before expiry respectively. This corresponds to very short-term and midterm maturities. Furthermore, the number of data point of the two sets was approximately equal. Comparing the estimates for the implied volatility and the option premiums of both methods, as well as the performance of the approximation with increasing maturity, we were able to show the superiority of the Δ-neutral method to the Vanna-Volga. The pronounced supremacy is although decreasing with increasing time to maturity.

We are interested in the goodness of both methods. The following results are valid for a fixed t. To focus on high vega strikes, the daily data set containing the received volatility estimates is vega-weighted. Vega takes its maximum for at-the-money options. A good approximation for the volatility in the region around $\mathfrak{m} = 1$ is more desirable, than those in the wings. Thus we introduce weighting coefficients for each strike $K_j \in \mathcal{K}_t$

$$\frac{\Lambda(K_j)}{\sum_{K_i \in \mathcal{K}_t} \Lambda(K_i)} \, .$$

Table 6.1 and Table 6.2 compare the methods for the first $(0.1205 \leq \tau \leq 0.0411)$ and the second $(0.2904 \leq \tau \leq 0.3699)$ data set respectively. Boxes with two columns give the corresponding data for both methods.

The *total volatility deviation* is computed by vega-weighting

$$\sum_{K_j \in \mathcal{K}_t} |\hat{\sigma}(K_j) - I(K_j)| \frac{\Lambda(K_j)}{\sum_{K_i \in \mathcal{K}_t} \Lambda(K_i)} \, . \tag{6.1}$$

Total premium deviation is the sum over deviations of premium estimates

$$\sum_{K_j \in \mathcal{K}_t} |\hat{C}(K_j) - C^{MK}(K_j)|$$

followed by the strike at which the *maximal premium deviation* was attained.
Maximal premium deviation

$$\max_{K_j} |\hat{C}(K_j) - C^{MK}(K_j)| \ .$$

Premium ratio is calculated for the maximal deviation

$$\frac{\hat{C}(K_j) - C^{MK}(K_j)}{C^{MK}(K_j)}$$

as well as the *estimated premium.*

Figure 6.1 and Figure 6.3, show the best estimates, that is those with minimal total volatility deviation, the premium estimates for the same set of strikes follow. Figure 6.2 as well as Figure 6.4 depicts the corresponding residuals:

the top graph: $(I - \hat{\sigma})$

the middle graph: $(C^{MK} - \hat{C})$

the bottom graph: $(C^{MK} - \hat{C})/C^{MK}$.

τ	*Total Volatility Deviation*		*Total Premium Deviation*		*Max. Premium attained at*		*Max. Premium Deviation*		*Premium Ratio*		*Estimated Premium*	
0.1205	0.00131	0.00123	26.6	15.6	3500	3800	3.0	1.7	0.00574	−0.00735	520.2	240.2
0.1178	0.00131	0.00119	26.4	13.7	3500	3200	2.8	1.6	0.00570	−0.00210	482.7	782.4
0.1151	0.00103	0.00086	28.0	20.8	3550	3200	3.4	3.0	0.00754	−0.00375	443.8	794.6
0.1068	0.00093	0.00068	28.4	11.3	3650	3250	3.2	1.3	0.00786	−0.00159	404.2	803.0
0.1041	0.00100	0.00084	22.3	13.5	3650	3300	2.6	1.9	0.00608	−0.00241	430.8	780.8
0.1014	0.00100	0.00081	22.2	13.6	3650	3300	2.7	2.0	0.00617	−0.00259	429.6	779.9
0.0986	0.00105	0.00088	22.6	9.8	3650	3650	2.7	1.0	0.00619	0.00216	428.8	430.6
0.0959	0.00107	0.00090	25.1	17.0	3650	3250	2.8	2.2	0.00655	−0.00272	420.5	821.0
0.0877	0.00093	0.00068	23.1	17.0	3700	3300	2.9	2.5	0.00737	−0.00315	394.9	795.6
0.0849	0.00113	0.00088	21.2	12.2	3700	3700	2.5	1.7	0.00654	0.00443	388.1	389.0
0.0822	0.00105	0.00087	19.1	14.1	3700	3300	2.2	1.7	0.00526	−0.00217	411.5	812.0
0.0795	0.00072	0.00063	18.4	12.3	3750	3300	2.1	1.5	0.00565	−0.00186	366.1	815.7
0.0767	0.00091	0.00061	21.2	18.7	3750	3300	2.7	2.3	0.00782	−0.00298	338.3	788.7
0.0685	0.00069	0.00048	18.1	14.5	3800	3300	2.4	1.8	0.00771	−0.00228	304.8	804.2
0.0658	0.00120	0.00105	17.2	17.1	3800	3350	2.3	2.0	0.00746	−0.00269	307.2	757.5
0.0630	0.00084	0.00046	18.7	21.0	3800	3300	2.5	2.4	0.00803	−0.00295	304.2	804.9
0.0603	0.00111	0.00072	17.1	22.4	3800	3300	2.5	2.7	0.00858	−0.00338	293.4	794.2
0.0575	0.00096	0.00077	19.8	17.0	3750	3250	2.4	1.8	0.00788	−0.00226	306.9	806.5
0.0493	0.00142	0.00089	22.4	59.9	3650	3600	2.7	5.8	0.00799	0.01490	339.9	385.6
0.0466	0.00212	0.00195	12.4	15.9	3600	3600	1.4	1.8	0.00382	0.00476	380.0	379.7
0.0438	0.00170	0.00170	9.9	7.0	3700	3700	1.2	0.6	0.00377	0.00194	327.2	327.8
0.0411	0.00246	0.00239	10.2	12.6	3700	3200	1.2	1.1	0.0040	−0.00144	295.0	794.0

Table 6.1: A comparison of methods for $0.1205 \leq \tau \leq 0.0411$. Blocks of two give the same characteristics for both methods. The first column contains data for Vanna-Volga method; the second for Δ-neutral. Data is given on daily basis. Volatility deviations computed by vega-weighting.

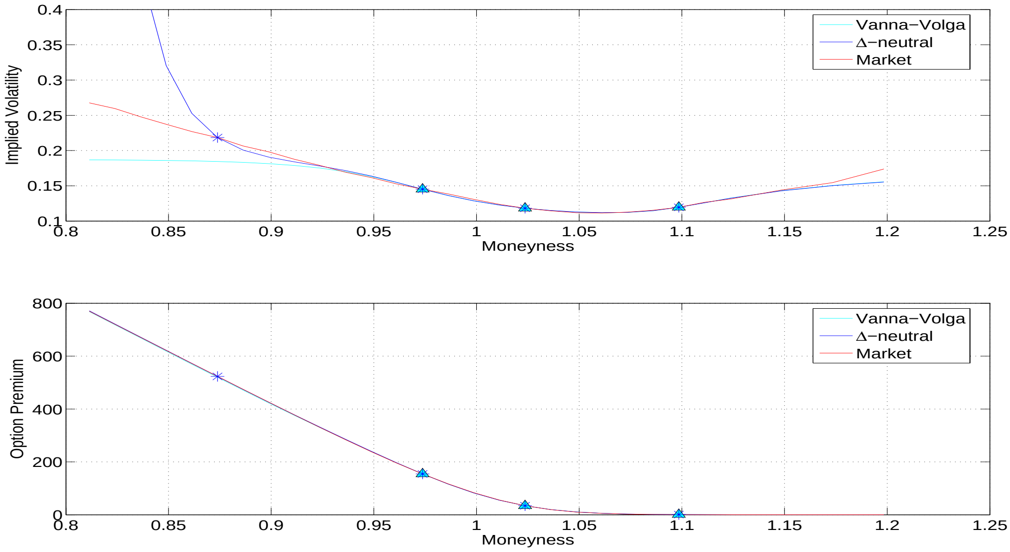

Figure 6.1: Best volatility and premium estimates OESX-1206, $\tau = 0.1205$
The upper graph shows the volatility approximations for Vanna-Volga (the light blue line) and its extension (the blue line) compared to the market implied volatility (red line), where the colored markers give the positions of the anker points.
The lower, the corresponding option premiums.

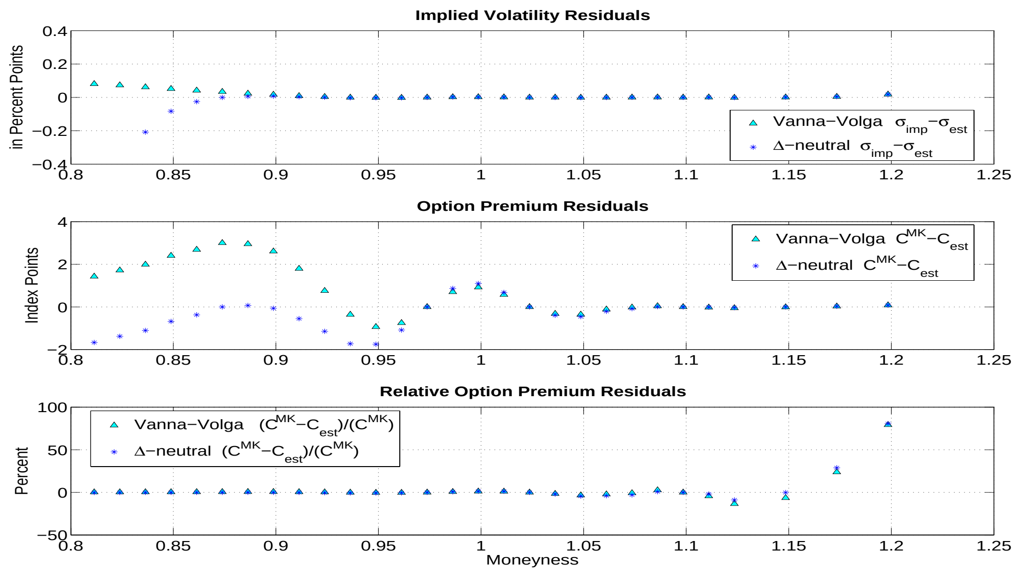

Figure 6.2: Volatility and premium residuals OESX-1206, $\tau = 0.1205$

τ	*Total Volatility Deviation*		*Total Premium Deviation*		*Max. Premium attained at*		*Max. Premium Deviation*		*Premium Ratio*		*Estimated Premium*	
0.3699	0.00140	0.00125	36.0	22.4	3200	4250	4.2	2.2	0.00497	−0.04465	842.4	50.5
0.3671	0.00151	0.00146	36.5	31.0	3200	3200	3.2	2.6	0.00401	0.00316	806.5	807.1
0.3644	0.00148	0.00145	35.2	36.3	3200	3200	3.6	4.2	0.00436	0.00518	816.3	815.7
0.3562	0.00124	0.00117	36.3	54.1	3250	3250	4.6	5.2	0.00560	0.00623	824.7	824.1
0.3534	0.00123	0.00116	41.3	41.7	3250	3250	4.0	3.9	0.00466	0.00451	851.7	851.8
0.3507	0.00136	0.00134	35.9	42.0	3250	3250	4.3	5.0	0.00499	0.00588	851.9	851.2
0.3479	0.00135	0.00131	34.5	27.8	3250	3700	4.2	2.3	0.00487	−0.00517	850.0	438.1
0.3452	0.00127	0.00123	34.2	24.8	3250	3250	3.7	1.8	0.00441	0.00214	842.5	844.4
0.3370	0.00102	0.00097	35.9	42.8	3300	3300	3.4	6.4	0.00419	0.00779	817.9	814.9
0.3342	0.00101	0.00096	24.6	21.9	3300	3300	3.7	2.4	0.00450	0.00298	809.9	811.2
0.3315	0.00097	0.00082	36.8	18.3	3300	3700	4.1	1.6	0.00521	−0.00338	784.4	464.7
0.3288	0.00078	0.00064	25.0	18.5	3300	3700	3.8	2.4	0.00453	−0.00513	836.5	468.8
0.3260	0.00110	0.00100	29.0	22.5	3300	3300	4.0	2.5	0.00495	0.00308	809.3	810.8
0.3178	0.00098	0.00076	32.4	45.0	3300	3300	4.2	6.8	0.00542	0.00818	775.8	821.3
0.3151	0.00115	0.00099	33.4	19.8	3300	3750	4.1	2.0	0.00529	−0.00487	777.9	412.1
0.3123	0.00105	0.00089	32.1	19.2	3300	3750	4.1	2.0	0.00530	−0.00479	777.2	412.1
0.3096	0.00099	0.00081	32.5	26.4	3300	4050	4.4	3.5	0.00565	0.02041	766.5	165.6
0.3068	0.00135	0.00122	36.3	28.5	3250	3650	4.7	2.9	0.00568	−0.00631	827.7	462.0
0.2986	0.00157	0.00141	37.8	28.0	3200	3950	4.6	2.0	0.00561	0.01134	812.6	174.3
0.2959	0.00158	0.00143	37.0	24.7	3200	3950	4.5	2.1	0.00559	0.01280	803.3	163.6
0.2932	0.00153	0.00150	36.1	32.2	3250	3250	4.1	2.9	0.00511	0.00364	799.7	800.9
0.2904	0.00145	0.00128	36.3	25.3	3200	3600	4.3	2.1	0.00526	−0.00470	815.9	449.4

Table 6.2: A comparison of methods for $0.2904 \leq \tau \leq 0.3699$. Blocks of two give the same characteristics for both methods. The first column contains data for Vanna-Volga method; the second for Δ-neutral. Data is given on daily basis. Volatility deviations computed by vega-weighting.

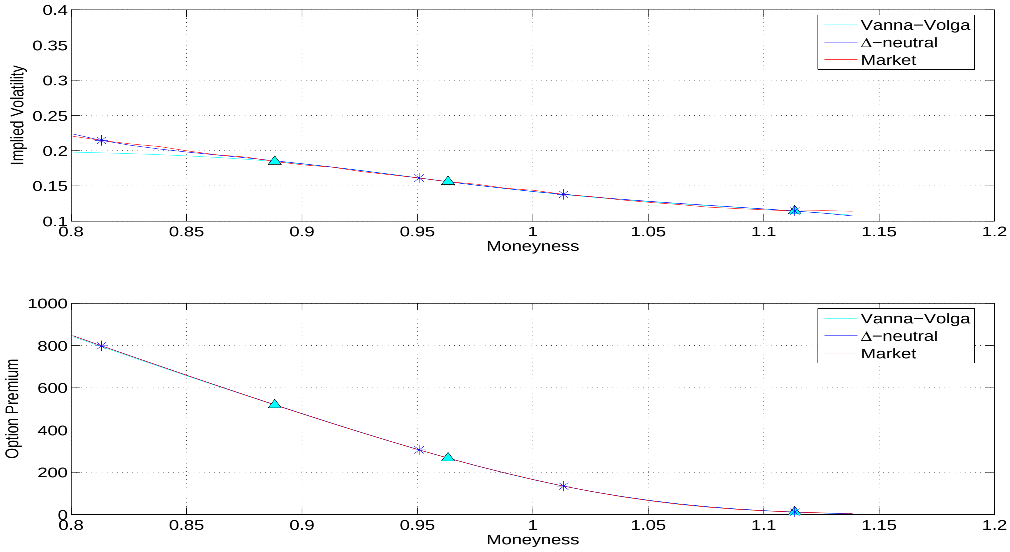

Figure 6.3: Best volatility and premium estimates OESX-0307, $\tau = 0.3699$.
The upper graph shows the volatility approximations for Vanna-Volga (the light blue line) and its extension (the blue line) compared to the market implied volatility (red line), where the colored markers give the positions of the anker points. The lower, the corresponding option premiums.

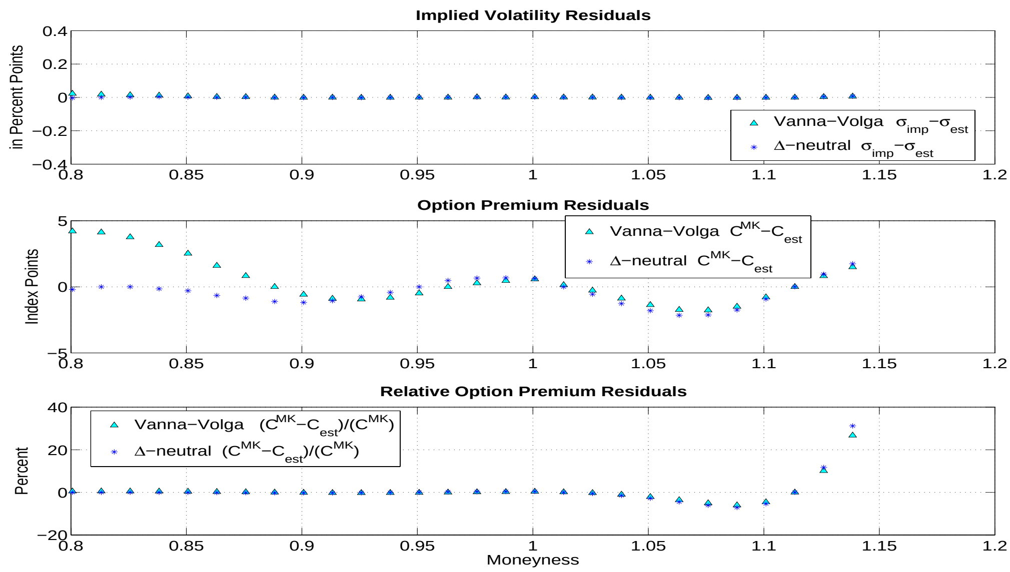

Figure 6.4: Volatility and premium residuals OESX-0307, $\tau = 0.3699$.

6.2 Discussion

As the first result we can point out that the Vanna-Volga method as well as its extension are applicable to equity index options.
Main results for both data sets are similar:

Over-all results become less precise, that is the accuracy region becoming more tight (compare Figure B.4), with declining maturity.

Out-of-the-money region

- Volatility approximation by both methods flattening.
- Premium approximation by Vanna-Volga is more precise – due to very small premiums, even tiny deviations have extreme consequences for the price and correspondingly to the ratio.

At-the-money region

- Volatility and premium approximation by both methods are very good.

In-the-money region

- Volatility approximation by Vanna-Volga extension is more precise – due to the fourth anker point, the moneyness range was extended by 0.1, which corresponds to approximately 8 strikes.
- Vanna-Volga extension produces much better premiums in absolute values; Due to the high premium prices for deep-in-the-money options, the ratios are approximately equal.

6.3 Choice of Anker Points

The choice of anker points in the Vanna-Volga method was driven by the fact, that the risk reversal and the vega-weighted butterfly belong to the few liquid options in the FX market. Surely, this choice does not deliver good results in all cases. The empirical results have shown, that the range of strikes becomes more tight with declining maturity – thus we are not able to give the set of strikes, delivering the best result for all maturities. Nevertheless there are regions delivering good results for almost all maturities (compare Figure B.3). Evaluating the corresponding histogram (Table A.3) we are able to give a recommendation for the choice of strikes in terms of moneyness:

$$K_1 \approx 0.875 \quad K_2 \approx 0.975 \quad K_3 \approx 1.025 \quad K_4 \approx 1.125\,. \tag{6.2}$$

Thus a pretty good choice would be equidistant distributed diametral strikes (compare Table A.4).

7 Pricing Under Stochastic Volatility

Though, the Vanna-Volga method is widely spread among traders, there is no mathematical explanation for the option pricing formula (5.9) – only heuristic justification with Itô exists. While searching for a theory which could explain the pricing formula, one particular model seemed to deliver interesting results: under mean-reverting volatility the appropriate option price could be represented by a Black-Scholes price adjusted by a sum of Gamma and Speed. Further investigation has shown, this model delivers the desired explanation.

7.1 Dynamics

We assume an underlying depending on volatility, which itself is a function of a mean-reverting Ornstein-Uhlenbeck (OU) process. Under a high rate of mean-reversion volatility is pulled back to its "natural" mean level on a shorter time scale than the remaining time to expiration of a particular option. The results of the first two sections were derived by Fouque et al. [7].

At first consider the dynamics:

$$S_t - S_0 = \int_0^t \mu S_u du + \int_0^t f(Y_u) S_u dW_u \tag{7.1}$$

$$Y_t - Y_0 = \int_0^t \alpha(m - Y_u) du + \int_0^t \beta d\hat{Z}_u \tag{7.2}$$

$$\hat{Z}_t := \rho W_t + \sqrt{1-\rho^2} Z_t$$

where W_t and Z_t are independent Brownian motions, α is the rate of mean reversion, m long run mean of Y_t, β is the volatility of volatility – *VolVol*, $|\rho| < 1$ is the correlation coefficient between price and volatility shocks,* $f(\cdot)$ some positive function. At discrete

*The case $\rho = 0$ implies smile effect, as shown by Renault and Touzi in [15].

times the price of the underlying is observable, volatility $\sigma_t := f(Y_t)$ is not observed directly and is subject to a hidden Markov process. The solution to (7.2) is

$$Y_t = m + \exp(-\alpha t)(Y_0 - m) + \beta \int_0^t \exp(-\alpha(t-u))d\hat{Z}_u$$

and given Y_0, Y_t is Gaussian

$$Y_t - \exp(-\alpha t)Y_0 \sim \mathcal{N}\left(m(1-\exp(-\alpha t)), \frac{\beta^2}{2\alpha}(1-\exp(-2\alpha t))\right).$$

The unique invariant distribution for Y is then $\mathcal{N}(m, \frac{\beta^2}{2\alpha})$ (see [10]), providing a simple building-block for stochastic volatility models with arbitrary $f(\cdot)$. The existence of a unique invariant distribution means, that Y is pulled towards its mean value m and the volatility of (7.1) approximately towards $f(m)$ as $t \to \infty$. In distribution it is the same as if α, the rate of mean reversion, tends to infinity.

7.2 Pricing

The following risk-neutral pricing is also valid for non-Markovian models.
By Girsanov's Theorem we introduce independent Brownian motions under an equivalent martingale measure $\mathbf{Q}^\lambda$

$$\begin{aligned} W_t^* :=& W_t + \int_0^t \frac{\mu - r}{f(Y_u)} du, \\ Z_t^* :=& Z_t + \int_0^t \lambda_u du \end{aligned} \tag{7.3}$$

assuming $(\frac{\mu-r}{f(Y_t)}, \lambda_t)$ satisfy the Novikov's condition. Since the market is incomplete (volatility is assumed to be a non-fungible asset; compare the Complete Market Theorem and the "Meta-theorem" in [1]) we denote this inability to derive a unique equivalent martingale measure by the dependance of $\mathbf{Q}$ on the *market price of volatility risk* λ; $\frac{\mu-r}{f(Y_t)}$ is called *excess return-to-risk ratio*. The approach here is that the derivative should be priced in order not to introduce any arbitrage into the market, – thus according to (3.6) and that the market, that is supply and demand, selects the unique equivalent martingale measure represented by λ to price derivatives.

Under new measure with the Radon-Nikodym derivative given by

$$\frac{d\mathbf{Q}^{\lambda}}{d\mathbf{P}} = \exp\left(-\frac{1}{2}\int_0^T \left(\frac{(\mu-r)^2}{f(Y_u)^2} + (\lambda_u)^2\right)du - \int_0^T \frac{\mu-r}{f(Y_u)}dW_u - \int_0^T \lambda_u dZ_u\right)$$

the equations (7.1) and (7.2) are written:

$$S_t - S_0 = \int_0^t rS_u du + \int_0^t f(Y_u)S_u dW_u^* \tag{7.4}$$

$$Y_t - Y_0 = \int_0^t \left[\alpha(m - Y_u) - \beta\left(\rho\frac{(\mu-r)}{f(Y_u)} + \lambda_u\sqrt{1-\rho^2}\right)\right]du + \int_0^t \beta d\hat{Z}_u^* \tag{7.5}$$

$$\hat{Z}_t^* := \rho W_t^* + \sqrt{1-\rho^2}Z_t^* .$$

Any allowable choice of λ leads to an equivalent martingale measure and by the Feynman-Kač Stochastic Representation Formula the option premium V writes as:

$$V_t = \mathbf{E}^{\mathbf{Q}^{\lambda}}\Big(\exp(-r\tau)h(S_T)|\mathcal{F}_t\Big) , \tag{7.6}$$

where $h(x)$ denotes the derivative payoff function.
If $\lambda = \lambda(t, S_t, Y_t)$ the setting is Markovian. In following we derive the partial differential equation corresponding this case.

Since the market lacks enough underlyings to price options in terms of those, the price of a particular derivative is not completely determined by the dynamics of its underlying and the requirement that the market is arbitrage-free. Thus a valuation with (7.6) requires a benchmark option G. Let G be an option with same parameters as V, but with a different strike:

$$V_t = v(t, S_t, Y_t), \quad \text{with} \quad V_T = (S_T - K)^+ ,$$
$$G_t = g(t, S_t, Y_t), \quad \text{with} \quad V_T = (S_T - K')^+ , \quad K \neq K' .$$

The price for V should satisfy market internal consistency relations, in order not to introduce any arbitrage opportunities. Taking the price of the benchmark option as a priory given, the prices of other derivatives are then uniquely determined – in consistency with the Meta-theorem.

A riskless portfolio consists now of two options and the underlying:

$$\Pi = V - \Delta^1 S - \Delta^2 G ,$$

which change over a time interval $\hat{t}$ is

$$\int_0^{\hat{t}} d\Pi_u = \int_0^{\hat{t}} dV_u - \int_0^{\hat{t}} \Delta^1_u dS_u - \int_0^{\hat{t}} \Delta^2_u dG_u \,. \tag{7.7}$$

Applying the Itô's Lemma on v and g, substituting from (7.1) and recombining the terms (7.7) becomes

$$\begin{aligned}\int_0^{\hat{t}} d\Pi_u =& \int_0^{\hat{t}} \left(\partial_t v + \frac{1}{2} f(Y_u)^2 S_u^2\, \partial_{ss} v + \frac{1}{2}\beta^2\, \partial_{yy} v + \rho f(Y_u) S_u \beta\, \partial_{sy} v\right) du \\ &- \int_0^{\hat{t}} \Delta^2_u \left(\partial_t g + \frac{1}{2} f(Y_u)^2 S_u^2\, \partial_{ss} g + \frac{1}{2}\beta^2\, \partial_{yy} g + \rho f(Y_u) S_u \beta\, \partial_{sy} g\right) du \\ &+ \int_0^{\hat{t}} (\partial_s v - \Delta^2_u\, \partial_s g - \Delta^1_u)\, dS_u + \int_0^{\hat{t}} (\partial_y v - \Delta^2_u\, \partial_y g)\, dY_u \,.\end{aligned} \tag{7.8}$$

A choice

$$\begin{aligned}\Delta^1 &= \partial_s v - \frac{\partial_s g\, \partial_y v}{\partial_y g}\,, \\ \Delta^2 &= \frac{\partial_y v}{\partial_y g}\end{aligned} \tag{7.9}$$

makes the portfolio risk-free, eliminating the integrands of dS_t and dY_t.

At the same time the portfolio earns at a risk-free rate r in absence of arbitrage opportunities:

$$\int_0^{\hat{t}} d\Pi_u = \int_0^{\hat{t}} r\Pi_u du.$$

A substitution from (7.9) and the above consideration lead to:

$$\begin{aligned}&\left(\partial_t v + \frac{1}{2} f(y)^2 s^2\, \partial_{ss} v + \frac{1}{2}\beta^2\, \partial_{yy} v + \rho f(y) s\beta\, \partial_{sy} v - rv + rs\, \partial_s v\right) / (\partial_y v) \\ =& \left(\partial_t g + \frac{1}{2} f(y)^2 s^2\, \partial_{ss} g + \frac{1}{2}\beta^2\, \partial_{yy} g + \rho f(y) s\beta\, \partial_{sy} g - rg + rs\, \partial_s g\right) / (\partial_y g)\ \ ,\end{aligned} \tag{7.10}$$

where v, g and their derivatives are evaluated at (t, S_t, Y_t).
Each side of (7.10) depends only on v or g respectively. Thus, both sides should be equal to some option-independent function (for we also could have taken an option g', similar

to v, but with a different maturity, instead of a different strike)

$$-\gamma(y) := \alpha(m-y) - \beta\left(\rho\frac{\mu-r}{f(y)} + \lambda(t,s,y)\sqrt{1-\rho^2}\right), \tag{7.11}$$

where $\lambda(t,s,y)$ is an arbitrary function.

The model parameters α, m, β, ρ, μ and λ are not constant in general. But identifying intervals of underlying stationarity we are able to take the parameters as constant. The price of volatility risk is determined solemnly by the benchmark option G, that is by the market itself.

The PDE corresponding to (7.6) is then written as

$$\partial_t v + \frac{1}{2}f(y)^2 s^2\,\partial_{ss}v + \frac{1}{2}\beta^2\,\partial_{yy}v + \rho f(y)s\beta\,\partial_{sy}v - rv + rs\,\partial_s v + \gamma(t,s,y)\,\partial_y v = 0 \tag{7.12}$$

subject to the terminal condition $v(T,s,y) = h(s)$.[†]

The rate of mean-reversion α is crucial for validating the applicability of the asymptotic analysis. Thus we are to prove the volatility of the DJ EURO STOXX 50 to be fast mean-reverting.[‡]

A research by Dotsis et al. (see [5][§]) explores several models describing the dynamics of implied volatility of the main American and European volatility indexes – among them VSTOXX. The estimation period covers a time span from 4/01/1999 to 24/03/2004. The likelihood function is estimated by the maximum-likelihood method from the density function of the process following (7.2).

An estimation of the volatility parameters states that volatility of EURO STOXX 50 is well described by a mean-reverting Gaussian process as in (7.2). Although other models, especially those, based on a jump diffusion model, produced better explanations for the time series, the estimates of the parameters α, m and β for a simple mean-reverting Gaussian process are still significant. This allows us to apply the asymptotic analysis described above.

Derived from the Black-Scholes price V^{BS} described in (3.5), with the underlying following the dynamics as in (3.4) and the volatility $\overline{\sigma}$ given by constant volatility, the following formula corrects the Black-Scholes price by a term containing the option Γ and

[†] For further details on the PDE see [19] and [17].

[‡] *Fast* mean-reverting in terms of the lifetime of the option, *slow* mean-reverting compared to the intraday data.

[§] A compact version of the paper is to appear in the *Journal of Banking and Finance*, lacking three models and some indexes, presented in the preprint.

Υ (compare Table 3.1)

$$V^c(t, S_t, \overline{\sigma}) = V^{BS}(t, S_t, \overline{\sigma}) - \tau \cdot H(t, S_t, \overline{\sigma}) \,, \tag{7.13}$$

where

$$H(t, S_t, \overline{\sigma}) = c_2 S_t^2 \frac{\partial^2 V^{BS}}{\partial s^2}(t, S_t, \overline{\sigma}) + c_3 S^3 \frac{\partial^3 V^{BS}}{\partial s^3}(t, S_t, \overline{\sigma}) \tag{7.14}$$

with c_2 and c_3 being constants related to the model parameters α, m, β, ρ and the functions f and λ. Containing information about the market, the coefficients c_2 and c_3 are not specific to any contract.

The coefficients are given by

$$\begin{aligned} c_2 :=& \overline{\sigma} \left(\big(\overline{\sigma} - b\big) - a\big(r + \frac{3}{2}\overline{\sigma}^2\big) \right) \,, \\ c_3 :=& - a\overline{\sigma}^3 \,. \end{aligned}$$

The estimates for a and b are derived from least-squares fitting to a linear function

$$I(t, S_t; K, T) = \hat{a} \left(\frac{\ln \mathfrak{m}}{\tau} \right) + \hat{b} \,.$$

where $I(t, S_t; K, T)$ are the implied volatilities of liquid near-the-money European call options of various strikes and maturities.

The variable $\left(\frac{\ln \mathfrak{m}}{\tau}\right)$ is referred to as *log-moneyness-to-maturity-ratio*, which states that volatility for longer maturities is linear function. This is often referred to as a *skew* and is observable in a market; Right before the expiration volatility is commonly U-shaped (compare Figure B.1).

7.3 Application on the Pricing Formula

Now we can derive the pricing formula from Section 5.2 under the assumption of the mean-reverting stochastic volatility.

Proposition 3

The choice of $x_i(t; K)$ as in (5.1) implies speed neutrality.

Proof
By the construction of the portfolio observe

$$\Lambda(t;K) = \sum_{i=1}^{4} x_i(t;K) \cdot \Lambda(t;K_i)$$
$$\frac{\partial \Lambda(t;K)}{\partial s} = \sum_{i=1}^{4} \frac{\partial x_i(t;K)}{\partial s} \cdot \Lambda(t;K_i) + \sum_{i=1}^{4} x_i(t;K) \cdot \frac{\partial \Lambda(t;K_i)}{\partial s}$$

and by vanna-neutrality

$$0 = \sum_{i=1}^{4} \frac{\partial x_i(t;K)}{\partial s} \cdot \Lambda(t;K_i)$$

which applying (5.2) yields

$$0 = \tau \sigma S^2 \cdot \left(\sum_{i=1}^{4} \frac{\partial x_i(t;K)}{\partial s} \cdot \Gamma(t;K_i) \right)$$
$$0 = \sum_{i=1}^{4} \frac{\partial x_i(t;K)}{\partial s} \cdot \Gamma(t;K_i) \,. \tag{7.15}$$

By vega-gamma neutrality for portfolios of European plain vanilla options stated in (5.2) it holds, that

$$\Gamma(t;K) = \sum_{i=1}^{4} x_i(t;K) \cdot \Gamma(t;K_i)$$

which differentiated with respect to S becomes

$$\Upsilon(t;K) = \sum_{i=1}^{4} \frac{\partial x_i(t;K)}{\partial s} \cdot \Gamma(t;K_i) + \sum_{i=1}^{4} x_i(t;K) \cdot \Upsilon(t;K_i) \,.$$

The assertion follows then with (7.15). ■

We form a hypothesis, that with λ being an independent parameter, the model described in Section 7.1 can be adjusted to match $C^{MK}(t;K_j)$ for all $K_j \in \mathcal{K}_t$. In other words, the market selects a unique equivalent martingale measure to price the derivatives and provides us with appropriate prices observable in the market. The value of market's price of volatility risk can thus be seen only in derivatives prices. This viewpoint is called *selecting an approximating complete market*. The consistency of (5.9) with the stochastic mean-reverting volatility model follows then by:

$$\begin{aligned}
\hat{C}(t;K;\overline{\sigma}) &= C^{BS}(t;K;\overline{\sigma}) + \sum_{i=1}^{4} x_i(t;K)\cdot[C^{MK}(t;K_i;\sigma_i) - C^{BS}(t;K_i;\overline{\sigma})] \\
&= C^{BS}(t;K;\overline{\sigma}) + \sum_{i=1}^{4} x_i(t;K)\cdot[C^{BS}(t;K_i;\overline{\sigma}) - \tau H(t;K_i;\overline{\sigma}) - C^{BS}(t;K_i;\overline{\sigma})] \\
&= C^{BS}(t;K;\overline{\sigma}) - \tau\sum_{i=1}^{4} x_i(t;K)H(t;K_i;\overline{\sigma}) \\
&= C^{BS}(t;K;\overline{\sigma}) - \tau\cdot H(t;K;\overline{\sigma})
\end{aligned} \tag{7.16}$$

with the last step following from the choice of x_i and Proposition 3.
That is, we replace the adjustment option to the Black-Scholes price in (7.13) with weighted sensitivities of liquid options.

8 Evaluation

The Vanna-Volga method as well as its extension are applicable to index options to adjust for a skew as well as for a smile. Both adjustments represent real extra and interpolation formulas, reproducing exactly the inputs. Although, neither the original method, nor the extension guarantee for convex premiums.

The valuating procedure is for instance applicable on illiquid deep in-the-money options, whose premiums need to be known for the evaluation of volatility indexes. For the same reason, one could use it for longer dated maturities, especially in the wings, since those are not covered by the obligations for market makers and thus are poorly traded. This could enable an extension of volatility-subindexes out to 5 years.

The formula for option prices (5.9), as well as the approximation of the implied volatility (5.13) both are easily implementable requiring no sophisticated algorithm and thus no special software for their derivation – a simple excel sheet would suffice.

A Tables

Date	Expiration 200612		Expiration 200703	
	Market	OLS-Forward	Market	OLS-Forward
11/01/2006	4022	4022	4051	4051.2
11/02/2006	3983	3983	4012	4011.9
11/03/2006	3994	3994	4023	4023.0
11/06/2006	4054	4054	4083	4083.0
11/07/2006	4081	4081	4111	4110.2
11/08/2006	4080	4080	4110	4110.5
11/09/2006	4079	4079	4109	4108.6
11/10/2006	4071	4071	4100	4100.5
11/13/2006	4095	4095	4125	4124.5
11/14/2006	4088	4088	4116	4116.8
11/15/2006	4112	4112	4141	4140.7
11/16/2006	4116	4116	4145	4144.7
11/17/2006	4088	4088	4116	4116.3
11/20/2006	4104	4104	4132	4132.0
11/21/2006	4107	4107	4135	4134.5
11/22/2006	4104	4104	4133	4133.6
11/23/2006	4093	4093	4122	4122.6
11/24/2006	4056	4056	4085	4084.6
11/27/2006	3989	3989	4018	4017.4
11/28/2006	3980	3980	4008	4008.5
11/29/2006	4027	4027	4056	4055.6
11/30/2006	3994	3994	4023	4022.5

Table A.1: Comparison of market futures prices with obtained forward prices.

Strike	x_1	x_2	x_3	x_4
3250	1.03244127	−0.06798949	0.06747687	−0.12158149
3300	1.03213298	−0.06728704	0.06669937	−0.12006348
3350	1.03106641	−0.06490269	0.06412412	−0.11513322
3400	1.02785631	−0.05788665	0.05676569	−0.10135903
3450	1.01941976	−0.039943	0.03861244	−0.06825608
3500	*1*	*2.43E − 17*	*−4.33E − 18*	*1.46E − 17*
3550	0.96075955	0.07737073	−0.07045937	0.11991063
3600	0.89106768	0.20742045	−0.17948611	0.296661
3650	0.78228683	0.39575872	−0.31885102	0.5078738
3700	0.63336433	0.62710768	−0.45650514	0.69437209
3750	0.45546439	0.85970073	−0.53792578	0.77247959
3800	0.27197992	1.03291294	−0.50370776	0.67259018
3850	0.1120851	1.08906021	−0.319133	0.38777758
3900	*1.03E − 16*	*1*	*0*	*−9.17E − 17*
3950	−0.05450027	0.78325375	0.38330196	−0.34199154
4000	−0.05899325	0.49692847	0.73004275	−0.48872944
4050	−0.03333633	0.21548568	0.95022704	−0.36362382
4100	*6.92E − 17*	*−6.92E − 17*	*1*	*2.31E − 17*
4150	0.02465001	−0.12143942	0.89321655	0.47874992
4200	0.03417112	−0.1555234	0.68654465	0.92008895
4250	0.03057972	−0.13109841	0.45022459	1.20690971
4300	0.02002475	−0.08193401	0.24123217	1.29346233
4350	0.0085912	−0.03386264	0.08974422	1.2030917
4400	*−8.66E − 18*	*1.73E − 17*	*0*	*1*
4450	−0.00466712	0.01739399	−0.04025493	0.75558854
4500	−0.00607199	0.02214908	−0.04893456	0.52487838
4600	−0.00416339	0.01467601	−0.03026727	0.20291469
4700	−0.00167121	0.00573973	−0.01127501	0.06047165
4800	−0.00047885	0.00161136	−0.00305251	0.01432504

Table A.2: Typical set of coefficients. The anker points are set in *italic*.

$0.2904 \leq \tau \leq 0.3699$					$0.2904 \leq \tau \leq 0.3699$					Cumulative				
$\mathfrak{m}$	K_1	K_2	K_3	K_4	$\mathfrak{m}$	K_1	K_2	K_3	K_4	$\mathfrak{m}$	K_1	K_2	K_3	K_4
0.805	1031	0	0	0	0.805	2919	0	0	0	0.805	3950	0	0	0
0.814	1645	0	0	0	0.815	4467	60	0	0	0.815	6112	60	0	0
0.823	1020	0	0	0	0.825	3893	257	0	0	0.825	5573	257	0	0
0.832	856	0	0	0	0.835	1963	281	3	0	0.835	3055	281	3	0
0.841	1569	0	0	0	0.845	1755	410	0	0	0.845	3181	410	0	0
0.850	1918	0	0	0	0.855	1526	748	0	0	0.855	2691	748	0	0
0.859	821	0	0	0	0.865	3049	1732	0	0	0.865	5201	1732	0	0
0.868	1331	0	0	0	0.875	2449	2177	91	0	0.875	4850	2177	91	0
0.877	2401	0	0	0	0.885	1578	1016	51	0	0.885	3272	1016	51	0
0.887	2663	0	0	0	0.895	866	695	28	0	0.895	2630	700	28	0
0.896	795	5	0	0	0.905	645	176	1	0	0.905	2634	176	1	0
0.905	1989	0	0	0	0.915	757	2117	194	1	0.915	3270	2178	196	1
0.914	2513	61	2	0	0.925	108	1150	160	4	0.925	2205	1825	160	4
0.923	2097	675	0	0	0.935	62	2322	484	5	0.935	2055	4789	484	5
0.932	173	384	0	0	0.945	48	1182	342	4	0.945	213	2263	362	4
0.941	1820	2083	0	0	0.955	163	2710	263	2	0.955	1006	5356	279	2
0.950	1008	3727	36	0	0.965	50	3643	896	24	0.965	527	7663	1111	24
0.959	477	4020	215	0	0.975	71	1747	221	2	0.975	266	7500	1136	2
0.969	0	0	0	0	0.985	26	1970	1058	27	0.985	97	7110	2820	37
0.978	195	5753	915	0	0.995	2	118	374	6	0.995	5	2068	2734	11
0.987	71	5140	1762	10	1.005	0	0	0	0	1.005	0	0	0	0
0.996	3	1950	2360	5	1.015	1	446	3126	41	1.015	1	1800	8354	145
1.005	0	0	0	0	1.025	0	242	3131	30	1.025	4	1170	8446	549

Continued next page

$0.2904 \le \tau \le 0.3699$					$0.2904 \le \tau \le 0.3699$					Cumulative				
$\mathfrak{m}$	K_1	K_2	K_3	K_4	$\mathfrak{m}$	K_1	K_2	K_3	K_4	$\mathfrak{m}$	K_1	K_2	K_3	K_4
1.023	4	928	5315	519	1.045	0	355	3033	23	1.045	0	393	5664	3152
1.032	0	36	1642	1146	1.055	0	0	939	20	1.055	0	0	1044	93
1.042	0	209	1924	304	1.065	0	194	2636	123	1.065	0	209	4802	4010
1.051	0	38	2736	3202	1.075	0	73	1092	405	1.075	0	95	2246	4637
1.060	0	15	2166	3887	1.085	0	11	1499	1440	1.085	0	11	2152	6058
1.069	0	0	43	1861	1.095	0	18	929	3234	1.095	0	18	1103	6825
1.078	0	22	1111	2371	1.105	0	0	239	3264	1.105	0	0	274	3974
1.087	0	0	653	4618	1.115	0	0	400	4516	1.115	0	0	444	6188
1.096	0	0	174	3591	1.125	0	0	520	5674	1.125	0	0	520	6693
1.105	0	0	35	710	1.135	0	0	114	1174	1.135	0	0	153	1856
1.114	0	0	44	1672	1.145	0	0	0	0	1.145	0	0	0	501
1.124	0	0	0	1019	1.155	0	0	0	88	1.155	0	0	0	88
1.133	0	0	0	537	1.165	0	0	0	0	1.165	0	0	0	198
1.142	0	0	39	145	1.175	0	0	0	1818	1.175	0	0	0	1818
1.151	0	0	0	501	1.185	0	0	0	3582	1.185	0	0	0	3582
1.160	0	0	0	198	1.195	0	0	0	835	1.195	0	0	0	835

Table A.3: Distribution histogram for strikes in terms of moneyness for both sets of data (first two blocks) and the cumulation (last block). Each block gives the moneyness and the number of occurrences of the corresponding strike. Moneyness specifies the center of an interval.

Date	OESX-1206				OESX-0307			
	Best estimates		Recommendation		Best estimates		Recommendation	
	Volatility	Settlement	Volatility	Settlement	Volatility	Settlement	Volatility	Settlement
11/01/2006	0.00123320	15.6	0.00124504	15.2	0.00125111	22.4	0.00188050	40.4
11/02/2006	0.00118573	13.7	0.00122011	13.9	0.00146464	31.0	0.00197538	38.3
11/03/2006	0.00085674	20.8	0.00093684	15.6	0.00144936	36.3	0.00162744	36.6
11/06/2006	0.00068015	11.3	0.00099526	12.7	0.00116553	54.1	0.00143271	40.7
11/07/2006	0.00083615	13.5	0.00136064	10.8	0.00116432	41.7	0.00168389	38.7
11/08/2006	0.00080996	13.6	0.00147320	11.0	0.00133955	42.0	0.00183812	46.8
11/09/2006	0.00088162	9.8	0.00148903	10.9	0.00131471	27.8	0.00185527	40.1
11/10/2006	0.00090477	17.0	0.00119838	12.1	0.00122863	24.8	0.00173694	36.4
11/13/2006	0.00067526	17.0	0.00095943	12.8	0.00097429	42.8	0.00102209	27.5
11/14/2006	0.00087818	12.2	0.00129085	11.9	0.00096220	21.9	0.00115699	23.4
11/15/2006	0.00087285	14.2	0.00190713	14.0	0.00082121	18.3	0.00128137	32.6
11/16/2006	0.00063299	12.3	0.00159450	15.5	0.00063809	18.5	0.00120473	27.2
11/17/2006	0.00060913	18.7	0.00599765	69.9	0.00099930	22.5	0.00124581	24.2
11/20/2006	0.00048440	14.5	0.00369157	45.9	0.00075904	45.0	0.00107054	25.5
11/21/2006	0.00105295	17.1	0.00666984	74.4	0.00098561	19.8	0.00147207	31.0
11/22/2006	0.00046173	21.0	0.00637502	102.5	0.00088876	19.2	0.00139242	30.8
11/23/2006	0.00072028	22.4	0.00945704	234.1	0.00080913	26.4	0.00107509	25.7
11/24/2006	0.00077051	17.0	0.00551496	40.3	0.00121868	28.5	0.00147495	36.0
11/27/2006	0.00088939	59.8	0.00512044	27.8	0.00141102	28.0	0.00148879	28.4
11/28/2006	0.00194807	15.9	0.00395699	11.6	0.00143435	24.7	0.00173283	26.6
11/29/2006	0.00170057	7.0	0.00787175	33.1	0.00149990	32.2	0.00169594	30.4
11/30/2006	0.00239487	12.6	0.01366915	94.1	0.00128437	25.3	0.00168894	40.1

Table A.4: Deviations of the best estimates and the recommended set of strikes (as in 6.2) from the market data on daily basis for both expiries. Volatility deviations calculated by vega-weighting.

B Figures

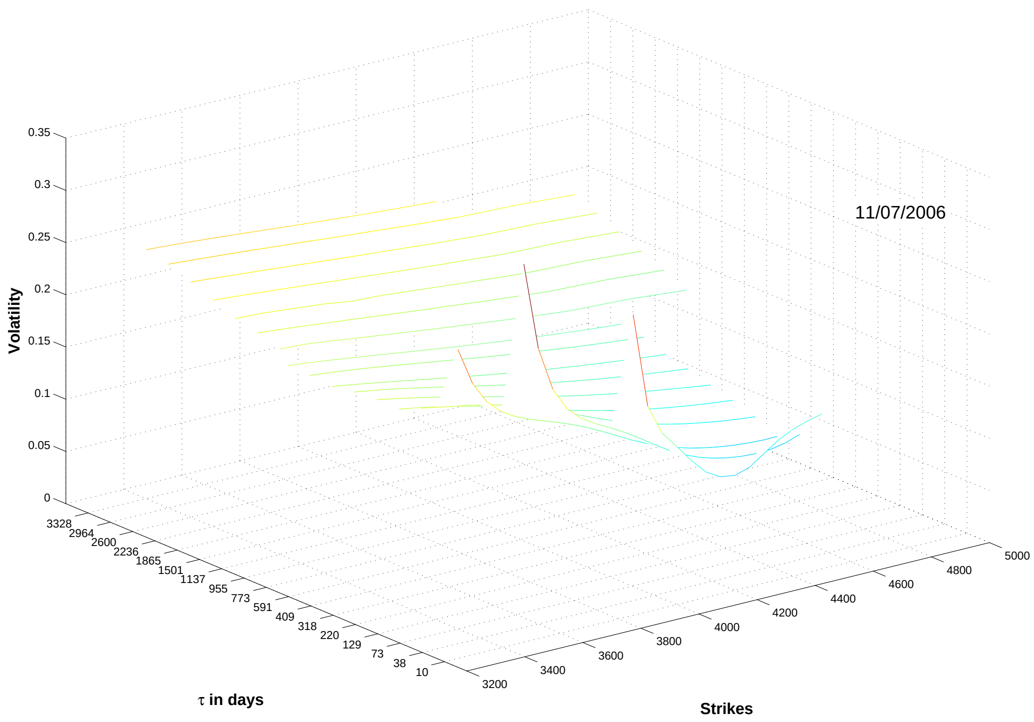

Figure B.1: Estimated volatility term structure.

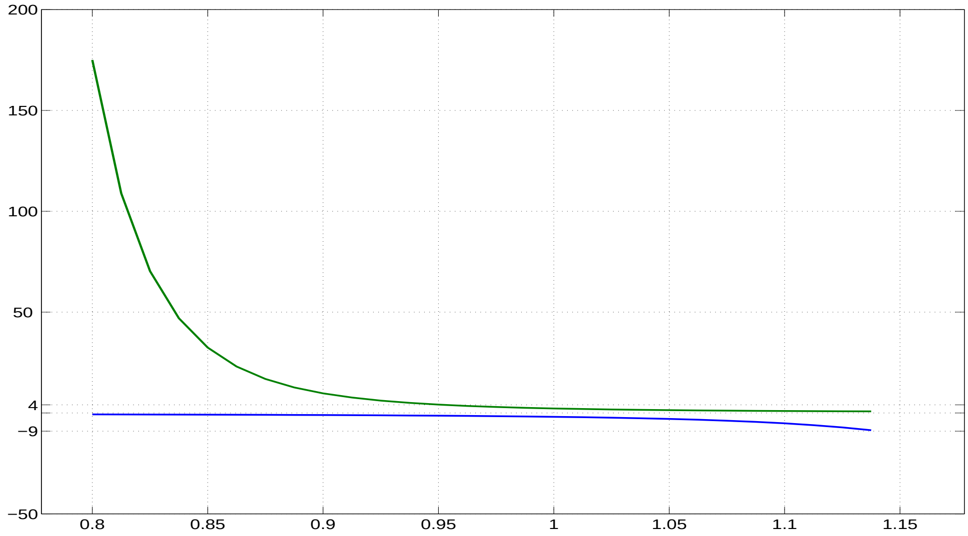

Figure B.2: Bounds for implied volatility slope OESX-0307, $\tau = 0.3699$.

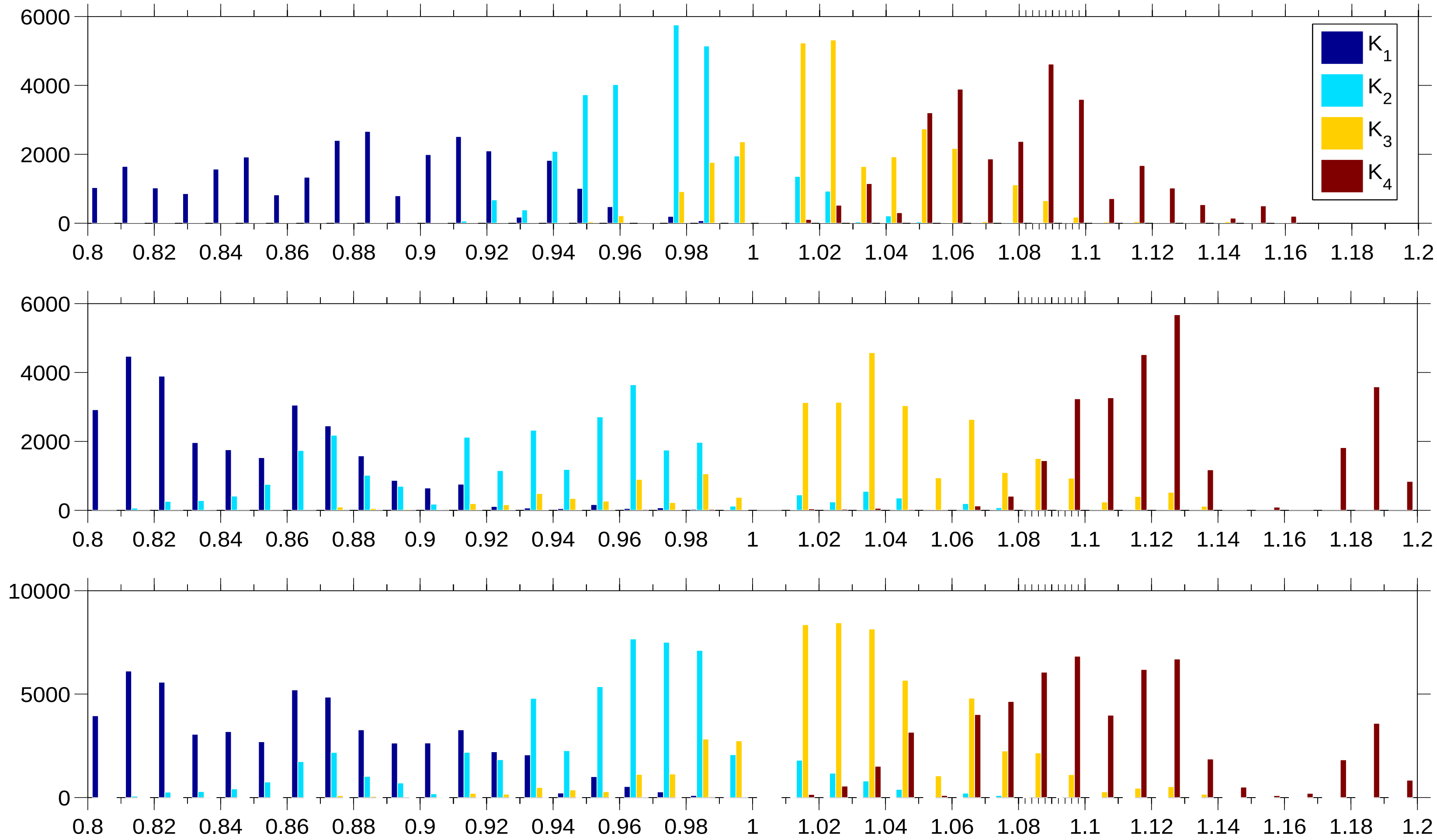

Figure B.3: Cumulative strikes distribution of the first 1200 sets per maturity.
Top – $0.1205 \leq \tau \leq 0.0411$, Middle – $0.2904 \leq \tau \leq 0.3699$, Bottom – cumulative for both expiries.

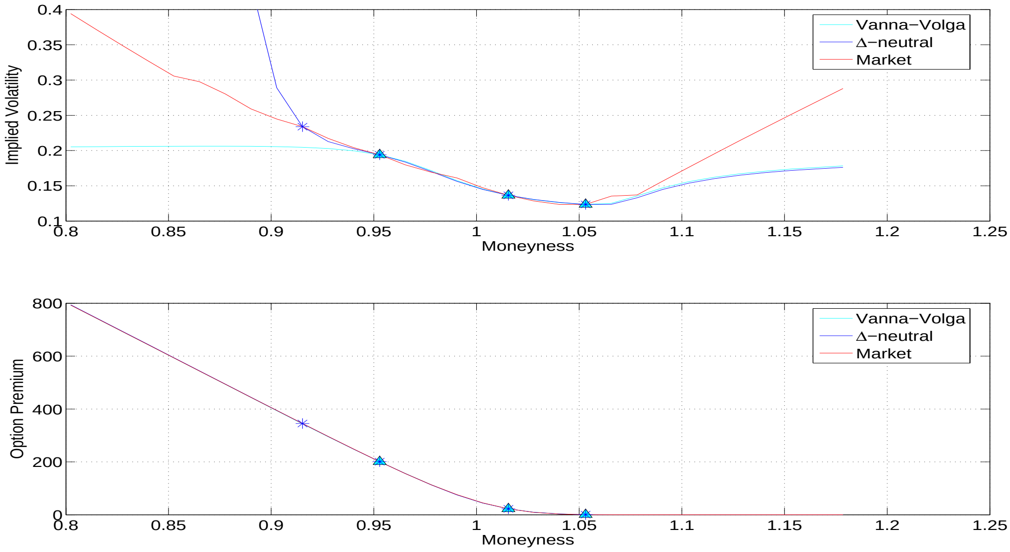

Figure B.4: Best volatility and premium estimates OESX-1206, $\tau = 0.0411$.
The upper graph shows the volatility approximations for Vanna-Volga (the light blue line) and its extension (the blue line) compared to the market implied volatility (red line), where the colored markers give the positions of the anker points.
The lower, the corresponding options' premiums.

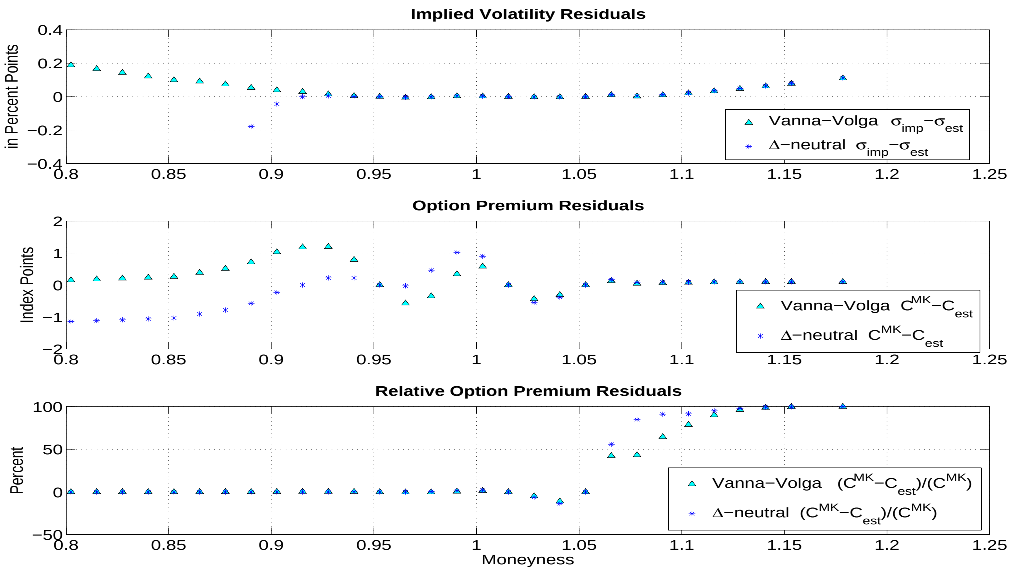

Figure B.5: Volatility and premium residuals OESX-1206, $\tau = 0.0411$.

C Matlab

A large part of the work was the implementation of the theoretical results in Matlab code. Here we give a summary of the Matlab procedure for a fixed t.
Matlab allows scalars as well as vectors as input. Scalars DATE, TAU, FUTURE and UNDERLYING (underlying close price) and vectors STRIKES, CALLS were easily received from the data delivered in matrix form – division by variables in vector form, taking to a power and multiplication have a special implementation:

The interest rate was derived in a variable RATE by the OLS-approximation according to (5.5)

```
RATE =@(X) sum((CALLS-PUTS+exp(-X*TAU).*(STRIKES-FUTURE)).^2) .
```

Next step was the calculation of Black-Scholes vectors Δ, Λ, Φ, Ξ with a constant reference volatility $\overline{\sigma}$. Financial toolbox relieve the handling with the Greeks – the most prominent of them are available:

```
blsprice, blsdelta, blsgamma, blsvega
```

giving the corresponding sensitivities.

To accomplish the calculation of volatility we had to check all $\binom{|\mathcal{K}_t|}{4}$ possibilities. A command

```
randerr(m,n,errors)
```

generates an m-by-n binary matrix, where errors determines how many nonzero entries are in each row. This gave us a matrix which applied to a vector, e.g. STRIKES, CALLS, Δ, Λ, Φ, Ξ, chose four variables as anker points. For all $\binom{|\mathcal{K}_t|}{4}$ combinations we calculated the coefficients $x_i(t;K)$, $i = \{1 \ldots 4\}$ subject to (5.8).

The calculation of the premium estimates according to (5.9) followed, which delivered with (5.13) the volatility estimates. As already mentioned the radicand in (5.13) is not always positive. Thus we had to filter the volatility estimates with an imaginary part, what reduced the number of daily combinations by more that a half. Vega-weighting as in (6.1) ordered the results.

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